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The microcanonical ensemble of the ideal relativistic quantum gas with angular momentum conservation

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arXiv:0707.0793v3 [nucl-th] 17 Oct 2007

Abstract

We derive the microcanonical partition function of the ideal relativistic quantum gas with fixed intrinsic angular momentum as an expansion over fixed multiplicities. We developed a group theoretical approach by generalizing known projection techniques to the Poincaré group. Our calculation is carried out in a quantum field framework and applies to particles with any spin. It extends known results in literature in that it does not introduce any large volume approximation and it takes particle spin fully into account. We provide expressions of the microcanonical partition function at fixed multiplicities in the limiting classical case of large volumes and large angular momenta and in the grand-canonical ensemble. We also derive the microcanonical partition function of the ideal relativistic quantum gas with fixed parity.

Key words:
PACS:

1 Introduction

In a previous work [1], we derived the microcanonical partition function of an ideal relativistic quantum gas of spinless bosons within a quantum field theoretical framework. Therein, we confined ourselves to the microcanonical ensemble with fixed energy and momentum. Here, we will obtain the partition function of the microcanonical ensemble of an ideal relativistic quantum gas with fixed energy-momentum *and* angular momentum. It should be pointed out from the very beginning that, for a quantum system, energy-momentum and angular momentum can be defined simultaneously only in the rest frame, i.e. if $\mathbf{P} = 0$. In fact, more rigorously, one should rather speak of fixing the maximal set of commuting observables of the orthochronous Poincaré group IO(1,3)$^{\uparrow}$, including energy-momentum and two observables formed out of the Pauli-Lubanski vector. However, for the sake of simplicity, we will retain the more colloquial expressions of “energy-momentum and angular momentum conservation” or “microcanonical ensemble with angular momentum conservation” being understood that we are working in the system rest frame, where the Pauli-Lubanski vector is proportional to the intrinsic angular momentum or spin of the system.

The microcanonical partition function of an ideal relativistic quantum gas with fixed angular momentum, in a full quantum field theoretical framework, has never been derived for particles with spin. It has been obtained in an essentially non-relativistic quantum mechanical framework in ref. [2] and an equivalent calculation has been performed in ref. [3] by using a projection method, yet only for spinless particles. With the same limitations, an expression with angular momentum conservation, but without momentum conservation has been derived by Koba [4]. The classical limit has been worked out in ref. [5]. In this work we will get over these limitations and derive the microcanonical partition function of the ideal relativistic quantum gas of particles with spin by using a group theoretical approach including the maximal set of observables pertaining to space-time symmetries, i.e. energy-momentum, spin, and parity, in a consistent way in a quantum field framework. Like in ref. [1], we will write it as an expansion over fixed multiplicities.

The most general expression will be used to work out special cases, such as the grand-canonical (fixed temperature) and the classical limit. We will show that those, more familiar, expressions, can be recovered, thus supporting the correctness of our approach.

Notation

In this paper we adopt the natural units, with $\hbar = c = K = 1$. Space-time linear transformations (translations, rotations, boosts) and SL(2,C) transformations are written in serif font, e.g. R, L. Operators in Hilbert space will be denoted by a hat, e.g. $\hat{\mathsf{R}}$. The only exception to this rule are field operators, which will be written as a capital Ψ, while c-numbers field functions with a small ψ. Also unit vectors will be denoted with an upper hat, but they will always be referred to explicitely in the text.

2 The projection onto irreducible Poincaré states

In order to calculate the microcanonical partition function of an ideal relativistic quantum gas fixing the maximal set of observables pertaining to space-time symmetries, the use of group theory is both essential and enlightening. The statement of the problem in this framework has been developed in refs. [6,7], generalizing known projection operator techniques used for internal, rather than space-time, symmetries, and we will outline it here.

In ref. [1] we showed that the microcanonical partition function of a relativistic system can be written as:

$$\Omega = \mathrm{e}^S = \sum_{h_V} \langle h_V | \mathsf{P}_i | h_V \rangle \ , \tag{1}$$

where $|h_V\rangle$ are a complete set of states for the localized problem (the quantum field in a compact region) and P_i is a projector operator which selects the quantum states with the fixed values of the physical observables; S is, by definition, the entropy. If only energy-momentum is involved, the relevant symmetry group is T(4) and the projector reads:

$$\mathsf{P}_i = \delta^4(P - \hat{P}) \tag{2}$$

Although $\delta^4(P - \hat{P})$ is not a proper projector because $\mathsf{P}^2 = a\mathsf{P}$ where a is a divergent constant [1] we will maintain this naming even for these non-idempotent operators relaxing mathematical rigour.

Similarly, for the maximal set of observables, the projection P_i should be made onto irreducible states i of the full symmetry group, namely the orthocronous Poincaré group IO(1,3)$^\uparrow$. These states can be built by diagonalizing operators or combinations of operators of the Poincaré algebra. The first Casimir is the squared four-momentum $\hat{P}^2$ with eigenvalue M^2. The construction of physical states depends on whether $M^2 > 0$ or $M^2 = 0$. We will consider only the former case, i.e. massive particles, and stick to the convention of ref. [8] for the definition of spin, that we will briefly outline.

First, the Pauli-Lubanski vector is formed:

$$\hat{W}_\mu = -(1/2) \sum_{\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} \hat{J}^{\nu\rho} \hat{P}^\sigma \tag{3}$$

$\hat{J}$ being the generators of the Lorentz group, with the following properties:

$$\begin{aligned} &[\hat{W}_\mu, \hat{P}_\nu] = 0 \\ &[\hat{W}_\mu, \hat{W}_\nu] = -i \sum_{\rho\sigma} \epsilon_{\mu\nu\rho\sigma} \hat{W}^\rho \hat{P}^\sigma \\ &\hat{W} \cdot \hat{P} = 0 \end{aligned} \tag{4}$$

[1] This is owing to the fact that normalized projectors onto irreducible representations cannot be defined for non-compact groups, such as the space-time translation group T(4)

In view of the first commutation relation, if $|p\rangle$ is an eigenvector of $\hat{P}$, so is $\hat{W}|p\rangle$. The restriction of $\hat{W}$ to the eigenspace labelled by four-momentum P is defined as $\hat{W}(P)$. Since $\hat{W}(P) \cdot P = 0$, this four-vector operator can be decomposed onto three orthonormal spacelike four-vectors $n_1(P), n_2(P), n_3(P)$ which form a basis of the Minkowski space with the unit vector $\hat{P} = P/\sqrt{(P^2)}$:

$$\hat{W}(P) = \sum_{i=1}^{3} \hat{W}_i(P) n_i(P) \tag{5}$$

It can be shown that the operators:

$$\hat{S}_i(P) = \hat{W}_i(P)/M \tag{6}$$

form an SU(2) algebra and are the suitable relativistic generalization of the spin angular momentum. The third component $\hat{S}_3(P)$ can be diagonalized along with $\hat{S}^2 = -\hat{W}^2/M^2$ which is a Casimir of the full group IO(1,3)$^\uparrow$, with corresponding eigenvalues λ and $J(J+1)$. With a suitable choice of $n_i(P)$, i.e.:

$$n_i(P) = [P]e_i \qquad [P] \equiv \mathsf{R}_3(\varphi)\mathsf{R}_2(\theta)\mathsf{L}_3(\xi) \tag{7}$$

e_i being the unit vectors of spacial axes and $[P]$ being a Lorentz transformation bringing the timelike vector $P_0 = (M, 0, 0, 0)$ into the four-momentum P with polar coordinates ξ, θ, φ; the eigenvalue λ has the physical meaning of the component of intrinsic angular momentum in the rest frame along the direction of particle momentum $\mathbf{P}$. Thus, with the choice (7), λ is the helicity in the rest frame and J is, by definition, *the* spin of the particle. Since, from eqs. (6), (5) and (3) $\hat{S}_i(P_0) = \hat{J}_i$, the spin operators in the rest frame coincide with the generators of the rotation groups.

The inclusion of parity adds another discrete quantum number Π to the maximal set of eigenvalues. Altogether, an irreducible state of the this group can be labelled by a four-momentum P, a spin J, its third component λ and parity Π and the projector onto such state P_i can then be written as:

$$\mathsf{P}_i = \mathsf{P}_{PJ\lambda\Pi} \tag{8}$$

Generalizing the known integral expression for compact groups, one can formally write the projector (8) as an integral over the Poincaré group:

$$\mathsf{P}_{PJ\lambda\Pi} = \frac{1}{2} \sum_{z=\mathsf{I},\mathsf{\Pi}} \dim \nu \int \mathrm{d}\mu(g_z)\, U^{\nu\dagger}_{ii}(g_z)\, \hat{g}_z \tag{9}$$

where μ is the invariant group measure, z is the identity or space inversion $\mathsf{\Pi}$, $g_z \in$ IO(1,3)$^\uparrow$, $U^\nu(g_z)$ is the matrix of the unitary irreducible representation ν the state i belongs to and $\hat{g}_z$ is the unitary representation of g_z in the Hilbert space. Like for the energy-momentum projector (2), the expression (9) is not, strictly speaking, a projector because the group IO(1,3)$^\uparrow$is non-compact and $\mathsf{P}^2_{PJ\lambda\Pi} = a\mathsf{P}_{PJ\lambda\Pi}$ with a divergent constant. However, we will keep calling (9) a projector relaxing mathematical rigor. Working in the rest frame of the system, with $P = (M, \mathbf{0})$, the matrix element $U^{\nu\dagger}_{ii}(g_z)$ vanishes unless Lorentz transformations are pure rotations and this implies the reduction of the integration in (9) from IO(1,3)$^\uparrow$ to $\mathrm{T}(4) \otimes \mathrm{SU}(2) \otimes \mathrm{Z}_2$ [6] (replacing SO(3) with its universal covering group SU(2)). The general transformation of the orthochronous Poincaré group g_z may be then factorized as:

$$g_z = \mathsf{T}(x)\mathsf{Z}\mathsf{\Lambda} = \mathsf{T}(x)\mathsf{Z}\mathsf{L}_{\hat{\mathbf{n}}}(\xi)\mathsf{R} \tag{10}$$

where $\mathsf{T}(x)$ is a translation by the four-vector x, Z is either the identity or the space inversion and $\mathsf{\Lambda} = \mathsf{L}_{\hat{\mathbf{n}}}(\xi)\mathsf{R}$ is a general orthochronous Lorentz transformation written as the product of a boost of hyperbolic angle ξ along the space-like axis $\hat{\mathbf{n}}$ and a rotation R depending on three Euler angles. Thus eq. (9) becomes:

$$\begin{aligned}
\mathsf{P}_{PJ\lambda\Pi} &= \frac{1}{2} \sum_{\mathsf{Z}=\mathsf{I},\mathsf{\Pi}} \frac{\dim\nu}{(2\pi)^4} \int \mathrm{d}^4x \int \mathrm{d}\mathsf{\Lambda}\; U^{\nu}_{ii}(\mathsf{T}(x)\mathsf{Z}\mathsf{\Lambda})^*\, \hat{\mathsf{T}}(x)\hat{\mathsf{Z}}\,\hat{\mathsf{\Lambda}} \\
&= \frac{1}{2} \sum_{\mathsf{Z}=\mathsf{I},\mathsf{\Pi}} \frac{\dim\nu}{(2\pi)^4} \int \mathrm{d}^4x \int \mathrm{d}\mathsf{\Lambda}\;\, \mathrm{e}^{iP\cdot x}\, \Pi^z U^{\nu}_{ii}(\mathsf{\Lambda})^*\; \hat{\mathsf{T}}(x)\hat{\mathsf{Z}}\,\hat{\mathsf{\Lambda}}
\end{aligned} \tag{11}$$

where $z = 0$ if $\mathsf{Z} = \mathsf{I}$ and $z = 1$ if $\mathsf{Z} = \mathsf{\Pi}$. In the above equation, the invariant measure d^4x of the translation subgroup has been normalized with a coefficient $1/(2\pi)^4$ in order to yield a Dirac delta, as shown later. Furthermore, $\mathrm{d}\mathsf{\Lambda}$ is the invariant normalized measure of the Lorentz group, which can be written as [9]:

$$\mathrm{d}\mathsf{\Lambda} = \mathrm{d}\mathsf{L}_{\mathbf{n}}(\xi)\,\mathrm{d}\mathsf{R} = \sinh^2\xi\mathrm{d}\xi\,\frac{\mathrm{d}\Omega_{\hat{\mathbf{n}}}}{4\pi}\,\mathrm{d}\mathsf{R} \tag{12}$$

$\mathrm{d}\mathsf{R}$ being the invariant measure of SU(2) group normalized to 1, $\xi \in [0, +\infty)$ and $\Omega_{\hat{\mathbf{n}}}$ are the angular coordinates of the unit vector $\hat{\mathbf{n}}$.

In the rest frame of the system, where $P = P_0 = (M, \mathbf{0})$, Lorentz transformations $\mathsf{\Lambda}$ comprising any non-trivial boost (i.e. with $\xi \neq 0$) would make the matrix element $U^{\nu}_{ii}(\mathsf{\Lambda})^*$ vanish. Therefore $\mathsf{\Lambda}$ reduces to the rotation R and (11) to:

$$\mathsf{P}_{PJ\lambda\Pi} = \frac{1}{2} \sum_{\mathsf{Z}=\mathsf{I},\mathsf{\Pi}} \frac{1}{(2\pi)^4} \int \mathrm{d}^4x\; (2J+1) \int \mathrm{d}\mathsf{R}\;\, \mathrm{e}^{iP\cdot x}\, \Pi^z\; D^J_{\lambda\lambda}(\mathsf{R})^*\, \hat{\mathsf{T}}(x)\,\hat{\mathsf{Z}}\,\hat{\mathsf{R}} \tag{13}$$

where we now use the ordinary symbol D to denote matrices of SU(2) unitary representations. The irreducible representations of this group are now labelled by eigenvalues of its generators $\hat{J}^2$ and $\hat{J}_3$ which coincide with the spin J and its third component λ defined above by means of operators $\hat{S}_i(P)$.

Taking into account that $[\mathsf{Z},\mathsf{R}]=0$, we can move the $\hat{\mathsf{Z}}$ operator to the right of $\hat{\mathsf{R}}$ and recast the above equation as:

$$\begin{aligned}\mathsf{P}_{PJ\lambda\Pi} &= \frac{1}{(2\pi)^4}\int \mathrm{d}^4x\ \mathrm{e}^{iP\cdot x}\,\hat{\mathsf{T}}(x)(2J+1)\int \mathrm{d}\mathsf{R}\ D^J_{\lambda\lambda}(\mathsf{R}^{-1})\,\hat{\mathsf{R}}\,\frac{\mathsf{I}+\Pi\hat{\Pi}}{2} \\ &= \delta^4(P-\hat{P})(2J+1)\int \mathrm{d}\mathsf{R}\ D^J_{\lambda\lambda}(\mathsf{R}^{-1})\,\hat{\mathsf{R}}\,\frac{\mathsf{I}+\Pi\hat{\Pi}}{2}\end{aligned} \tag{14}$$

Π being the parity of the system. The Eq. (14) is indeed the final general expression of the projector defining the proper microcanonical ensemble with $P=(M,\mathbf{0})$, where all conservation laws related to space-time symmetries are implemented. The nice feature of the above expression is the factorization of projections onto the energy-momentum P, spin-helicity J,λ and parity Π which allows us to calculate the contribution of angular momentum and energy-momentum conservation separately.

3 The full microcanonical partition function

The microcanonical partition function (MPF) (1) can be written also as, by inserting a resolution of identity $\sum_f |f\rangle\langle f|$:

$$\Omega = \sum_f \langle f|\mathsf{P}_i\mathsf{P}_V|f\rangle = \mathrm{tr}[\mathsf{P}_i\mathsf{P}_V] \tag{15}$$

where $\mathsf{P}_V \equiv \sum_{h_V}|h_V\rangle\langle h_V|$ is the projector onto localized states $|h_V\rangle$ and $\mathsf{P}_i=\mathsf{P}_{PJ\lambda\Pi}$ defined in Sect. 2. In ref. [1], we argued that P_V can be written, in its most general form, in a field theoretical framework as a functional integral over eigenstates $|\psi\rangle$ of the (scalar charged) quantum field:

$$\begin{aligned}&\mathsf{P}_V = \int_V \mathcal{D}(\psi,\psi^\dagger)|\psi,\psi^\dagger\rangle\langle\psi,\psi^\dagger| \\ &\text{with}\qquad |\psi,\psi^\dagger\rangle \equiv \otimes_{\mathbf{x}\in V}|\psi(\mathbf{x}),\psi^\dagger(\mathbf{x}')\rangle \otimes_{\mathbf{x}\notin V}|\psi(\mathbf{x}),\psi^\dagger(\mathbf{x}')\rangle\end{aligned} \tag{16}$$

where the field function $\psi(\mathbf{x})$ is some arbitrary function outside the region V [2]. In the functional integration (16), the index V implies that integrated field variables are just those in V, i.e. $\mathcal{D}\psi=\prod_{\mathbf{x}\in V} d\psi(\mathbf{x})$. With this definition, spurious terms depending on the degrees of freedom outside V are introduced which must be subtracted "by hand" in a consistent way. This means that no contribution arising from these degrees of freedom should be retained at the end of the calculation [1].

[2] We will use the same symbol V to denote the system region and its volume.

The quantum field theoretical approach was suggested in ref. [7] to be essential for small volumes, because localized states are in principle a superposition of states with different number of particles (meant as asymptotic states of the free field defined over the whole space). The approximation of assuming localized states having a well-defined number of particles is expected to be a good one only when the size of the region is much larger than Compton wavelength. Yet, the final expression of the microcanonical partition function obtained in ref. [1] only differs by that calculated with the aforementioned approximation by an immaterial constant factor, after due subtraction of spurious terms. In this work, we will extend this result to particles with spin, that is to non-scalar fields.

As a first step, the MPF Ω is expressed as an expansion over all possible *channels*. By channel we mean a set of particle multiplicities for each species, that is a set of integers $N_1, \ldots, N_K \equiv \{N_j\}$ where K is the total number of species. In formula:

$$\Omega = \sum_{\{N_j\}} \Omega_{\{N_j\}} \ . \tag{17}$$

$\Omega_{\{N_j\}}$ is defined as the *microcanonical channel weight* and can be calculated integrating on kinematical degrees of freedom. Writing $|f\rangle \equiv |\{N_j\},\{k\}\rangle$ where by $\{k\}$ we label the set of kinematical variables (momenta and spins) of the free Fock space state $|f\rangle$, the microcanonical weight of the channel $\{N_j\}$ reads:

$$\Omega_{\{N_j\}} = \sum_{\{k\}} \omega_{\{N_j\},\{k\}} = \sum_{\{k\}} \langle \{N_j\},\{k\}|\mathsf{P}_i \mathsf{P}_V|\{N_j\},\{k\}\rangle \ . \tag{18}$$

where $\omega_{\{N_j\},\{k\}}$ is defined as the *microcanonical state weight*. It is now useful to insert a resolution of the identity in $\omega_{\{N_j\},\{k\}}$ by using again the completeness of a set of Fock space states $|\{N_j\}',\{k'\}\rangle$:

$$\omega_{\{N_j\},\{k\}} \equiv \sum_{\{N_j\}',\{k'\}} \langle \{N_j\},\{k\}|\mathsf{P}_i|\{N_j\}',\{k'\}\rangle \langle \{N_j\}',\{k'\}|\mathsf{P}_V|\{N_j\},\{k\}\rangle \tag{19}$$

and work out the matrix elements $\langle f|\mathsf{P}_i|f'\rangle$ and $\langle f'|\mathsf{P}_V|f\rangle$ separately. It is easy to realize that P_i in (19) cannot change the multiplicities of the ket $|f'\rangle$. This can be seen from its integral expression (14): translation, rotations and reflections cannot change the number of particles of the state vector they are acting upon. Therefore, in the sum in eq. (19) $N' = N$ and:

$$\omega_{\{N_j\},\{k\}} \equiv \sum_{\{k'\}} \langle \{N_j\},\{k\}|\mathsf{P}_i|\{N_j\},\{k'\}\rangle \langle \{N_j\},\{k'\}|\mathsf{P}_V|\{N_j\},\{k\}\rangle \tag{20}$$

We will start our derivation of the MPF by considering the projector $\mathsf{P}_i \equiv \mathsf{P}_{PJ\lambda}$ without parity fixing, which will be addressed in Sect. 10. According to eq. (14), the reduced projector $\mathsf{P}_{PJ\lambda}$ can be written as:

$$\mathsf{P}_{PJ\lambda} = \delta^4(P - \hat{P})(2J+1) \int \mathrm{d}\mathsf{R}\; D^J_{\lambda\lambda}(\mathsf{R}^{-1})\, \hat{\mathsf{R}} \; . \tag{21}$$

We will first work out the channel with one particle and calculate Ω_1. This will serve as a basis for the calculation of the general microcanonical channel weight in Sec. 5.

4 Single particle channel

Henceforth, we will stick to the notation of ref. [8]. Single particle states are written as $|[p],\sigma\rangle$ and fulfill:

$$\hat{P}|[p],\sigma\rangle = p|[p],\sigma\rangle \qquad \text{and} \qquad \hat{S}_3(p)|[p],\sigma\rangle = \sigma|[p],\sigma\rangle \tag{22}$$

with normalization:

$$\langle [p],\sigma|[q],\tau\rangle = \delta^3(\mathbf{p}-\mathbf{q})\delta_{\sigma\tau} \; . \tag{23}$$

With $[p]$ we denote the SL(2,C) (universal covering group of $\mathrm{SO}(1,3)^{\uparrow}_{+}$) matrix corresponding to the Lorentz transformation transforming $p_0 = (m, \mathbf{0})$ into $p = (\varepsilon, \mathbf{p})$. This is similar to the notation of eq. (7), with the difference that now the notation is extended to SL(2,C) and expressions like $[p]^\dagger$ are now meaningful. In the previous equations, square brackets in $|[p],\tau\rangle$ warns us that, strictly speaking, the corresponding state (and also the operator $\hat{S}_3(p)$) has been defined relatively to a non-unique set of axes $p, n_i(p);\; i = 1,2,3$ (see Sect. 2), but this will not affect our calculation in any way [8] and we will drop henceforth this notation letting:

$$|[p],\sigma\rangle = |p,\sigma\rangle \; . \tag{24}$$

The transformation of a state $|p,\sigma\rangle$ under a general Lorentz transformation $\mathsf{\Lambda}$ reads:

$$\hat{\mathsf{\Lambda}}|p,\sigma\rangle = \sum_\tau |\mathsf{\Lambda}p,\tau\rangle D^S_{\tau\sigma}([\mathsf{\Lambda}p]^{-1}\mathsf{\Lambda}[p]) \sqrt{\frac{(\mathsf{\Lambda}p)^0}{p^0}} \; . \tag{25}$$

where S is the spin of the particle, $(\mathsf{\Lambda}p)^0$, p^0 are respectively the time component of $\mathsf{\Lambda}p$, p and $[\mathsf{\Lambda}p]^{-1}\mathsf{\Lambda}[p]$ is the Wigner rotation. A special case of (25) occurs when $\mathsf{\Lambda}$ is a rotation:

$$\hat{\mathsf{R}}|p,\sigma\rangle = \sum_\tau |\mathsf{R}p,\tau\rangle D^S_{\tau\sigma}([\mathsf{R}p]^{-1}\mathsf{R}[p]) \; . \tag{26}$$

According to eq. (20), we have to evaluate:

$$\omega_{1,(p\sigma)} \equiv \sum_{p',\sigma'} \langle p,\sigma|\mathsf{P}_i|p',\sigma'\rangle\langle p',\sigma'|\mathsf{P}_V|p,\sigma\rangle \; . \tag{27}$$

By using the integral expansion of the projector $\mathsf{P}_{PJ\lambda}$ in eq. (21), we can easily provide an integral expression of the first matrix element in eq. (27):

$$\langle p,\sigma|\mathsf{P}_{PJ\lambda}|p',\sigma'\rangle = \delta^4(P-p)(2J+1)\int \mathrm{d}\mathsf{R}\; D^J_{\lambda\lambda}(\mathsf{R}^{-1}) \\ \times \delta^3(\mathsf{R}\mathbf{p}'-\mathbf{p})D^S_{\sigma\sigma'}([\mathsf{R}p']^{-1}\mathsf{R}[p']) \; . \tag{28}$$

Conversely, it is much more difficult to develop the expression of the second matrix element $\langle p,\sigma|\mathsf{P}_V|p,\sigma'\rangle$. For spinless particles, this requires a functional method, described in detail in ref. [1]. To cope with this problem, we first need the expression of fields associated to particles with generic spin S. The free fields corresponding to the $2(2S+1)$ physical degrees of freedom of a particle and antiparticle with spin S in Heisenberg representation can be written as [8] (see also ref. [10]):

$$\begin{aligned}\Psi_\tau(x) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\sum_\tau \int \frac{\mathrm{d}^3\mathrm{p}}{\sqrt{2\varepsilon}}\; D^S_{\tau\sigma}([p])a(p,\sigma)\,\mathrm{e}^{-ip\cdot x} + D^S_{\tau\sigma}([p]C^{-1})b^\dagger(p,\sigma)\,\mathrm{e}^{ip\cdot x}\\ \tilde{\Psi}_\tau(x) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\sum_\tau \int \frac{\mathrm{d}^3\mathrm{p}}{\sqrt{2\varepsilon}}\; D^S_{\tau\sigma}([p]^{\dagger-1})a(p,\sigma)\,\mathrm{e}^{-ip\cdot x} + D^S_{\tau\sigma}([p]^{\dagger-1}C)b^\dagger(p,\sigma)\,\mathrm{e}^{ip\cdot x}\end{aligned} \tag{29}$$

where $\varepsilon = \sqrt{\mathrm{p}^2+m^2}$ is the energy; a, $a^\dagger$ and b, $b^\dagger$ are respectively destruction and creation operators for particles and antiparticles; D^S is the finite-dimensionsal representation of SL(2,C) of dimension $2S+1$ and the SL(2) matrix $C \equiv i\sigma_2$ (σ_i being a Pauli matrix). The (anti)commutation relations of creation and destruction operators in (29) are related to the normalization of states (23) and read:

$$[a(p,\sigma),a^\dagger(p',\sigma')]_\pm = [b(p,\sigma),b^\dagger(p',\sigma')]_\pm = \delta^3(\mathbf{p}-\mathbf{p}')\delta_{\sigma\sigma'} \; . \tag{30}$$

Under a Lorentz (or SL(2,C)) transformation, the fields in eq. (29) transform according to [8,10]:

$$\begin{aligned}\hat{\Lambda}\Psi_\tau(x)\hat{\Lambda}^{-1} &= \sum_\sigma D^S_{\tau\sigma}(\Lambda^{-1})\Psi_\sigma(\Lambda x)\\ \hat{\Lambda}\tilde{\Psi}_\tau(x)\hat{\Lambda}^{-1} &= \sum_\sigma D^S_{\tau\sigma}(\Lambda)^\dagger\tilde{\Psi}_\sigma(\Lambda x) \; .\end{aligned} \tag{31}$$

where we have written them in the Heisenberg representation. Note that $D^S(\Lambda)$ and $D^S(\Lambda)^{\dagger-1}$ are two inequivalent finite-dimensional non-unitary representation of the Lorentz group and correspond to those labelled as $(S,0)$ and $(0,S)$ respectively. As representations of SL(2,C) they have useful properties:

$$D^S(A^\dagger) = D^S(A)^\dagger \qquad D^S(A^T) = D^S(A)^T \; . \tag{32}$$

The matrix $C = i\sigma_2$ fulfills following relations:

$$C = -C^T = C^* \qquad C^2 = -1 \qquad CAC^{-1}A^T = (\det A)\, I \ \ \forall A \in \mathrm{SL}(2,\mathrm{C}) \tag{33}$$

and plays a crucial role in the spin-statistics connection, as:

$$D^S_{\tau\sigma}(C^2) = (-1)^{2S}\delta_{\tau\sigma} \ . \tag{34}$$

The $2(2S+1)$ field degrees of freedom in eq. (29) are needed to represent particle and antiparticle. Since $\tilde{\Psi}_\tau(x) = D^S_{\tau\sigma}(C)\Psi^{c\dagger}_\sigma(x)$; where Ψ^c is the charge-conjugated field which is obtained from Ψ by swapping $a \leftrightarrow b$ and $a^\dagger \leftrightarrow b^\dagger$ in the eq. (29), for a particle coinciding with its antiparticle, $\tilde{\Psi}_\tau(x) = D^S_{\tau\sigma}(C)\Psi^\dagger_\sigma(x)$, a constraint which effectively halves the field degrees of freedom.

It is easily seen from (29) that for equal times $t = t'$ $[\Psi(\mathbf{x}), \tilde{\Psi}(\mathbf{x}')]_\pm = 0$, where the sign $+$ or $-$ applies to the fermion or boson case respectively, so it is possible to put together the fields Ψ and $\tilde{\Psi}$ in one spinor $\boldsymbol{\Psi}$ with $2(2S+1)$ independent (anti)commuting components:

$$\boldsymbol{\Psi} = \begin{pmatrix} \Psi \\ \tilde{\Psi} \end{pmatrix} \tag{35}$$

and write the two fields in eq. (29) in a compact way as:

$$\boldsymbol{\Psi}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_\sigma \int \frac{\mathrm{d}^3\mathrm{p}}{\sqrt{2\varepsilon}}\ U(p,\sigma)a(p,\sigma)\,\mathrm{e}^{-ip\cdot x} + V(p,\sigma)b^\dagger(p,\sigma)\,\mathrm{e}^{ip\cdot x} \tag{36}$$

where $U(p,\sigma)$ and $V(p,\sigma)$ are $2(2S+1)$-dimensional spinors:

$$U(p,\sigma) = \begin{pmatrix} D^S_{1,\sigma}([p]) \\ \vdots \\ D^S_{2S+1,\sigma}([p]) \\ D^S_{1,\sigma}([p]^{\dagger-1}) \\ \vdots \\ D^S_{2S+1,\sigma}([p]^{\dagger-1}) \end{pmatrix} \qquad V(p,\sigma) = \begin{pmatrix} D^S_{1,\sigma}([p]C^{-1}) \\ \vdots \\ D^S_{2S+1,\sigma}([p]C^{-1}) \\ D^S_{1,\sigma}([p]^{\dagger-1}C) \\ \vdots \\ D^S_{2S+1,\sigma}([p]^{\dagger-1}C) \end{pmatrix} \tag{37}$$

It is also possible to write the field in an even more compact way defining the $2(2S+1)\times(2S+1)$ matrices $U(p)$ and $V(p)$ by clumping the spinors $U(p,\sigma)$ and $V(p,\sigma)$ respectively:

$$U(p) = \begin{pmatrix} D^S([p]) \\ D^S([p]^{\dagger-1}) \end{pmatrix} \qquad V(p) = \begin{pmatrix} D^S([p]C^{-1}) \\ D^S([p]^{\dagger-1}C) \end{pmatrix} \tag{38}$$

so that the eq. (36) can be recast as:

$$\Psi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{\mathrm{d}^3\mathrm{p}}{\sqrt{2\varepsilon}} \left(U(p)a(p)\,\mathrm{e}^{-ip\cdot x} + V(p)b^\dagger(p)\,\mathrm{e}^{ip\cdot x} \right) \tag{39}$$

$a(p)$ and $b^\dagger(p)$ being $2S+1$-dimensional column vectors of destruction and creation operators. Now, let us define the $(2S+1) \times 2(2S+1)$ conjugate spinorial matrices:

$$\bar{U}(p) = U^\dagger(p)\Gamma_0 \qquad \bar{V}(p) = V^\dagger(p)\Gamma_0 \tag{40}$$

Γ_0 being the $2(2S+1) \times 2(2S+1)$ matrix:

$$\Gamma_0 = \begin{pmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{pmatrix} .$$

The following relations apply:

$$\begin{aligned} &\bar{U}(p)U(p) = (-1)^{2S}\bar{V}(p)V(p) = 2\mathrm{I} \\ &\bar{U}(p)V(p) = \bar{V}(p)U(p) = \begin{cases} 2D^S(C) & \text{if } S \,\text{even} \\ 0 & \text{if } S \,\text{odd} \end{cases} \end{aligned} \tag{41}$$

as well as:

$$\begin{aligned} U(p)\bar{U}(p) &= \begin{pmatrix} \mathrm{I} & D^S([p][p]^\dagger) \\ D^S([p][p]^\dagger)^{-1} & \mathrm{I} \end{pmatrix} \\ V(p)\bar{V}(p) &= \begin{pmatrix} (-1)^{2S}\mathrm{I} & D^S([p][p]^\dagger) \\ D^S([p][p]^\dagger)^{-1} & (-1)^{2S}\mathrm{I} \end{pmatrix} \end{aligned} \tag{42}$$

One can construct the conjugate field $\bar{\Psi}$:

$$\bar{\Psi}(x) \equiv \Psi^\dagger(x)\Gamma_0 = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{\mathrm{d}^3\mathrm{p}}{\sqrt{2\varepsilon}} \left(a^\dagger(p)\bar{U}(p)\,\mathrm{e}^{ip\cdot x} + b(p)\bar{V}(p)\,\mathrm{e}^{-ip\cdot x} \right) \tag{43}$$

The fields Ψ and $\bar{\Psi}$ obey homogenous generalized Dirac equations for spin S [8,10]. Particularly, for $S = 1/2$ it can be shown that $\sqrt{(m)}\Psi$ is the usual Dirac field and Ψ obeys the familiar Dirac equations in the Weyl representation:

$$i\gamma^\mu \partial_\mu \Psi - m\Psi = 0 \qquad \gamma^0 = \Gamma_0 \;\; \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} . \tag{44}$$

To proceed with the calculation of the matrix element $\langle p,\sigma|\mathsf{P}_V|p,\sigma'\rangle$ we now have to express the creation and destruction operators as a function of the fields. We do this by using the fields in Schrödinger representation, i.e. with the fields at $t=0$:

$$\begin{aligned}
\langle 0|a(p,\sigma) &= \langle 0|\ \frac{1}{2(2\pi)^{\frac{3}{2}}}\sum_\tau\int \mathrm{d}^3\mathrm{x}\ \mathrm{e}^{-i\mathbf{p}\cdot\mathbf{x}}\sqrt{2\varepsilon}\,\bar{U}(p)_{\sigma\tau}\Psi_\tau(\mathbf{x}) \\
a^\dagger(p,\sigma)|0\rangle &= \frac{1}{2(2\pi)^{\frac{3}{2}}}\sum_\tau\int \mathrm{d}^3\mathrm{x}\ \mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}}\sqrt{2\varepsilon}\,\bar{\Psi}_\tau(\mathbf{x})U(p)_{\tau\sigma}\ |0\rangle \\
\langle 0|b(p,\sigma) &= \langle 0|\ \frac{1}{2(2\pi)^{\frac{3}{2}}}\sum_\tau\int \mathrm{d}^3\mathrm{x}\ \mathrm{e}^{-i\mathbf{p}\cdot\mathbf{x}}\sqrt{2\varepsilon}\,\bar{\Psi}_\tau(\mathbf{x})V(p)_{\tau\sigma}(-1)^{2S} \\
b^\dagger(p,\sigma)|0\rangle &= \frac{1}{2(2\pi)^{\frac{3}{2}}}\sum_\tau\int \mathrm{d}^3\mathrm{x}\ \mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}}\sqrt{2\varepsilon}\,\bar{V}(p)_{\sigma\tau}\Psi_\tau(\mathbf{x})(-1)^{2S}\ |0\rangle\ ;
\end{aligned} \tag{45}$$

these relations can be checked on the basis of (41). Therefore, using (45) for particles:

$$\begin{aligned}
&\langle p',\sigma'|\mathsf{P}_V|p,\sigma\rangle = \langle 0|a(p',\sigma')\mathsf{P}_V a^\dagger(p,\sigma)|0\rangle \\
&= \frac{2\sqrt{\varepsilon\varepsilon'}}{4(2\pi)^3}\int \mathrm{d}^3\mathrm{x}'\int \mathrm{d}^3\mathrm{x}\ \mathrm{e}^{-i\mathbf{p}'\cdot\mathbf{x}'+i\mathbf{p}\cdot\mathbf{x}}\sum_{\tau,\tau'}\bar{U}(p')_{\sigma'\tau'}\langle 0|\Psi_{\tau'}(\mathbf{x}')\mathsf{P}_V\bar{\Psi}_\tau(\mathbf{x})|0\rangle U(p)_{\tau\sigma}
\end{aligned} \tag{46}$$

The difficult task in the above equation is to calculate the vacuum expectation value in presence of the projector P_V. For the scalar case, we could cope with this problem through a functional integration of the field degrees of freedom in the region V and dropping the terms involving the points outside the region V [1]. However, the functional method developed for the scalar field cannot be trivially carried over to the non-scalar case for a twofold reason: firstly, even in the bosonic case, the fields Ψ and $\bar{\Psi}$ do not commute at equal times and, as a consequence, writing an expansion of the operator P_V like in eq. (16) is not possible; secondly, one has to deal with fermionic degrees of freedom.

Yet we have some clues of how to work out (46), getting inspiration from the scalar case. The presence of P_V means that eventually we have to restrict all spacial integrations to the region V. Then, we will *assume* that:

$$\int_V \mathrm{d}^3\mathrm{x}'\ \mathrm{e}^{-i\mathbf{p}'\cdot\mathbf{x}'}\langle 0|\Psi_{\tau'}(\mathbf{x}')\mathsf{P}_V\bar{\Psi}_\tau(\mathbf{x})|0\rangle = \frac{\mathrm{e}^{-i\mathbf{p}'\cdot\mathbf{x}}}{2\varepsilon'}\,(U(p')\bar{U}(p'))_{\tau'\tau}\langle 0|\mathsf{P}_V|0\rangle \tag{47}$$

and:

$$\int_V \mathrm{d}^3\mathrm{x}'\ \mathrm{e}^{-i\mathbf{p}'\cdot\mathbf{x}'}\langle 0|\bar{\Psi}_{\tau'}(\mathbf{x}')\mathsf{P}_V\Psi_\tau(\mathbf{x})|0\rangle = \frac{\mathrm{e}^{-i\mathbf{p}'\cdot\mathbf{x}}}{2\varepsilon'}\,(V(p')\bar{V}(p'))_{\tau'\tau}\langle 0|\mathsf{P}_V|0\rangle\ . \tag{48}$$

The justification of the formulae (47) and (48) resides in two facts:

- they are consistent with the limit $\mathsf{P}_V \to \mathsf{I}$, as it can be readily checked from the explicit field expression and the normalization condition (23);
- they have been proved for the scalar case [1].

If one accepts the conjecture expressed by the equations (47) and (48) the eq. (46) turns into, by using (41):

$$
\begin{aligned}
&\langle p', \sigma' | \mathsf{P}_V | p, \sigma \rangle \\
&= \frac{\langle 0 | \mathsf{P}_V | 0 \rangle}{4(2\pi)^3} \sum_{\tau\tau'} \sqrt{\frac{\varepsilon}{\varepsilon'}} \int_V \mathrm{d}^3\mathrm{x} \; \mathrm{e}^{-i\mathbf{x}\cdot(\mathbf{p}'-\mathbf{p})} \, \bar{U}(p')_{\sigma'\tau'} (U(p')\bar{U}(p'))_{\tau'\tau} U(p)_{\tau\sigma} \\
&= \frac{1}{2} \sqrt{\frac{\varepsilon}{\varepsilon'}} \, F_V(\mathbf{p}-\mathbf{p}') (\bar{U}(p') U(p))_{\sigma'\sigma} \langle 0 | \mathsf{P}_V | 0 \rangle \\
&= \frac{1}{2} \sqrt{\frac{\varepsilon}{\varepsilon'}} \, F_V(\mathbf{p}-\mathbf{p}') \left(D^S([p']^{-1}[p]) + D^S([p']^{\dagger}[p]^{\dagger -1}) \right)_{\sigma'\sigma} \langle 0 | \mathsf{P}_V | 0 \rangle
\end{aligned} \tag{49}
$$

where F_V is a Fourier integral over the system region V:

$$
F_V(\mathbf{p}-\mathbf{p}') = \frac{1}{(2\pi)^3} \int_V \mathrm{d}^3\mathrm{x} \; \mathrm{e}^{i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')} \; . \tag{50}
$$

For $p = p'$ and $\sigma = \sigma'$ the above expression reduces to $V/(2\pi)^3 \langle 0 | \mathsf{P}_V | 0 \rangle$ which is the same we obtained for the scalar case in ref. [1]. It is straightforward to show that the same expression holds for antiparticles.

We are now in a position to calculate the microcanonical state weight ω_1. By plugging (28) and (49) in the definition (27):

$$
\begin{aligned}
&\omega_1 = \\
&= \delta^4(P-p) \sum_{\sigma'} \int \mathrm{d}^3\mathrm{p}' \; (2J+1) \int \mathrm{d}\mathsf{R} \; D^J_{\lambda\lambda}(\mathsf{R}^{-1}) \delta^3(\mathsf{R}\mathbf{p}'-\mathbf{p}) D^S_{\sigma\sigma'}([\mathsf{R}p']^{-1}\mathsf{R}[p']) \\
&\times \frac{1}{2} \sqrt{\frac{\varepsilon}{\varepsilon'}} F_V(\mathbf{p}-\mathbf{p}') \left(D^S([p']^{-1}[p]) + D^S([p']^{\dagger}[p]^{\dagger -1}) \right)_{\sigma'\sigma} \langle 0 | \mathsf{P}_V | 0 \rangle \\
&= \delta^4(P-p) \sum_{\sigma'} \int \mathrm{d}^3\mathrm{p}' \; (2J+1) \int \mathrm{d}\mathsf{R} \; D^J_{\lambda\lambda}(\mathsf{R}^{-1}) \delta^3(\mathsf{R}\mathbf{p}'-\mathbf{p}) D^S_{\sigma\sigma'}([p]^{-1}\mathsf{R}[p']) \\
&\times \frac{1}{2} F_V(\mathbf{p}-\mathbf{p}') \left(D^S([p']^{-1}[p]) + D^S([p']^{\dagger}[p]^{\dagger -1}) \right)_{\sigma'\sigma} \langle 0 | \mathsf{P}_V | 0 \rangle
\end{aligned} \tag{51}
$$

Two matrix products are to be computed in the above equation. The first yields:

$$
\sum_{\sigma'} D^S_{\sigma\sigma'}([p]^{-1}\mathsf{R}[p']) D^S_{\sigma'\sigma}([p']^{-1}[p])) = D^S([p]^{-1}\mathsf{R}[p])_{\sigma\sigma} \tag{52}
$$

To work out the second matrix product we observe that $[\mathsf{R}p']^{-1}\mathsf{R}[p'] = [p]^{-1}\mathsf{R}[p']$ is a Wigner *rotation*, hence it is unitary:

$$
[p]^{-1}\mathsf{R}[p'] = ([p]^{-1}\mathsf{R}[p'])^{-1\dagger} = ([p']^{-1}\mathsf{R}^{-1}[p])^{\dagger} = ([p]^{\dagger}\mathsf{R}[p']^{\dagger -1})
$$

thus:

$$\sum_{\sigma'} D^S_{\sigma\sigma'}([p]^{-1}\mathsf{R}[p'])D^S_{\sigma'\sigma}([p']^{\dagger}[p]^{\dagger-1}) = \sum_{\sigma'} D^S_{\sigma\sigma'}([p]^{\dagger}\mathsf{R}[p']^{\dagger-1})D^S_{\sigma'\sigma}([p']^{\dagger}[p]^{\dagger-1})$$
$$= D^S_{\sigma\sigma}([p]^{\dagger}\mathsf{R}[p]^{\dagger-1}) \tag{53}$$

By using (52) and (53) the eq. (51) becomes:

$$\omega_1 = \delta^4(P-p)\int \mathrm{d}^3\mathrm{p}'\ (2J+1)\int \mathrm{d}\mathsf{R}\ D^J_{\lambda\lambda}(\mathsf{R}^{-1})\delta^3(\mathsf{R}\mathbf{p}'-\mathbf{p})$$
$$\times\frac{1}{2}F_V(\mathbf{p}-\mathbf{p}')\left(D^S([p]^{-1}\mathsf{R}[p]) + D^S([p]^{\dagger}\mathsf{R}[p]^{\dagger-1})\right)_{\sigma\sigma}\langle 0|\mathsf{P}_V|0\rangle \tag{54}$$

This is the final expression of the single-particle channel. We note in passing that, with the same method, the more general matrix element $\langle p,\sigma'|\mathsf{P}_i\mathsf{P}_V|p,\sigma\rangle$ could have been calculated, yielding:

$$\langle p,\sigma'|\mathsf{P}_i\mathsf{P}_V|p,\sigma\rangle = \delta^4(P-p)\int \mathrm{d}^3\mathrm{p}'\ (2J+1)\int \mathrm{d}\mathsf{R}\ D^J_{\lambda\lambda}(\mathsf{R}^{-1})\delta^3(\mathsf{R}\mathbf{p}'-\mathbf{p})$$
$$\times\frac{1}{2}F_V(\mathbf{p}-\mathbf{p}')\left(D^S([p]^{-1}\mathsf{R}[p]) + D^S([p]^{\dagger}\mathsf{R}[p]^{\dagger-1})\right)_{\sigma'\sigma}\langle 0|\mathsf{P}_V|0\rangle \tag{55}$$

5 Multiparticle channels

We first tackle the problem of a channel with N particles of different species. Writing the Fock state $|N,\{k\}\rangle$ in eq. (20) as:

$$|f\rangle = |N,\{k\}\rangle = |N,\{p_n\},\{\sigma_n\}\rangle$$

where $\{p_n\}$; $\{\sigma_n\}$ label the set of four-momenta and helicities of the N particles, the microcanonical state weight in eq. (20) becomes:

$$\omega_f = \sum_{\{\sigma'_n\}}\left[\prod_{n=1}^{N}\int \mathrm{d}^3\mathrm{p}'_n\right]\langle N,\{p_n\},\{\sigma_n\}|\mathsf{P}_{PJ\lambda}|N,\{p'_n\},\{\sigma'_n\}\rangle$$
$$\times\langle N,\{p'_n\},\{\sigma'_n\}|\mathsf{P}_V|N,\{p_n\},\{\sigma_n\}\rangle\ . \tag{56}$$

Both matrix elements of $\mathsf{P}_{PJ\lambda}$ and P_V can be calculated by a straightforward generalization of the single-particle channel because, as particles are distinct, creation and destruction operators, as well as field operators, commute with each other and the problem is fully factorizable. Therefore:

$$\langle N,\{p_n\},\{\sigma_n\}|\mathsf{P}_{PJ\lambda}|N,\{p'_n\},\{\sigma'_n\}\rangle = \delta^4\left(P-\sum_{n=1}^N p_n\right)$$
$$\times\,(2J+1)\int \mathrm{d}\mathsf{R}\; D^J_{\lambda\lambda}(\mathsf{R}^{-1})\left[\prod_{n=1}^N D^{S_n}_{\sigma_n\sigma'_n}([\mathsf{R}p'_n]^{-1}\mathsf{R}[p'_n])\delta^3(\mathsf{R}\mathbf{p}'_n-\mathbf{p}_n)\right] \quad (57)$$

and:

$$\langle N,\{p_n\},\{\sigma'_n\}|\mathsf{P}_V|N,\{p'_n\},\{\sigma_n\}\rangle = \langle 0|\mathsf{P}_V|0\rangle$$
$$\times\prod_{n=1}^N \frac{1}{2}\sqrt{\frac{\varepsilon_n}{\varepsilon'_n}}F_V(\mathbf{p}_n-\mathbf{p}'_n)\left(D^S([p'_n]^{-1}[p_n])+D^S([p'_n]^\dagger[p_n])^{\dagger-1}\right)_{\sigma'_n\sigma_n} \quad . \quad (58)$$

Putting (57) and (58) into (56) and working out the matrix products, like in eqs. (52),(53) we get:

$$\omega_f = \langle 0|\mathsf{P}_V|0\rangle\,(2J+1)\int \mathrm{d}\mathsf{R}\; D^J_{\lambda\lambda}(\mathsf{R}^{-1})\left[\prod_{n=1}^N\int \mathrm{d}^3\mathrm{p}'_n\;\delta^3(\mathsf{R}\mathbf{p}'_n-\mathbf{p}_n)\right.$$
$$\left.\times\,\frac{1}{2}\left(D^S([p_n]^{-1}\mathsf{R}[p_n])+D^S([p_n]^\dagger\mathsf{R}[p_n]^{\dagger-1})\right)_{\sigma_n\sigma_n}F_V(\mathbf{p}_n-\mathbf{p}'_n)\right]\delta^4\left(P-\sum_{n=1}^N p_n\right) \quad (59)$$

The integration over momenta p'_n can now be performed. Since $\mathbf{p}'_n=\mathsf{R}^{-1}\mathbf{p}_n$:

$$\omega_f = \langle 0|\mathsf{P}_V|0\rangle\,\delta^4\left(P-\sum_{n=1}^N p_n\right)(2J+1)\int \mathrm{d}\mathsf{R}\; D^J_{\lambda\lambda}(\mathsf{R}^{-1})$$
$$\times\left[\prod_{n=1}^N\frac{1}{2}\left(D^S([p_n]^{-1}\mathsf{R}[p_n])+D^S([p_n]^\dagger\mathsf{R}[p_n]^{\dagger-1})\right)_{\sigma_n\sigma_n}F_V(\mathbf{p}_n-\mathsf{R}^{-1}\mathbf{p}_n)\right] \quad (60)$$

which is the final expression of the microcanonical state weight for N different particles. Integrating (60) over momenta p_n and summing over spin projections σ_n, one gets the microcanonical channel weight in a simple form:

$$\Omega_N = (2J+1)\int \mathrm{d}\mathsf{R}\; D^J_{\lambda\lambda}(\mathsf{R}^{-1})\left[\prod_{n=1}^N\int \mathrm{d}^3\mathrm{p}_n\;\mathrm{tr}D^{S_n}(\mathsf{R})\right.$$
$$\left.\times\,F_V(\mathbf{p}_n-\mathsf{R}^{-1}\mathbf{p}_n)\right]\delta^4\left(P-\sum_{n=1}^N p_n\right)\langle 0|\mathsf{P}_V|0\rangle\;. \quad (61)$$

where we took advantage of cyclicity property of the trace. It is interesting to note that Wigner rotations and momentum-dependent matrices in the matrix element of P_V (58) combine so as to leave, once the sum over spin projections is carried out, a trace of a simple SU(2) rotation.

We will now deal with the more complicated case of identical particles.

5.1 *Identical particles*

Developing the matrix element $\langle f'|\mathsf{P}_V|f\rangle$ for a multiparticle state with identical particles is far more difficult than for different particle species because the associated fields do not commute and factorization does not hold; this has been shown for the scalar field in our previous work [1]. Nevertheless, we showed that, keeping only terms involving field degrees of freedom in the region V, the result is the same one would get in a multi-particle non-relativistic quantum mechanical framework enforcing symmetrization or antisymmetrization for bosons or fermions respectively. Although the conclusion was fair, the proof involved lenghty calculations which would become even lenghtier for particles with spin. Therefore, we will not tackle the proof here and simply assume that it holds for particles with spin: *a general multiparticle state can be treated as a state with different particles, with (anti)symmetrization for identical particles.* Accepting this conjecture allows us to work on the fixed-number multiparticle tensor space instead of the original Fock space.

Let ρ_j be a permutation of the integers $1, \dots, N_j$, $\chi(\rho_j)$ its parity and $b_j = 0, 1$ for bosons and fermions respectively. The generic state $|f\rangle$ in the multiparticle tensor space can be written as:

$$|f\rangle = \sum_{\{\rho_j\}} \left[\prod_{j=1}^{K} \frac{\chi(\rho_j)^{b_j}}{\sqrt{N_j!}} \right] \prod_{j=1}^{K} |N_j, \{p_{\rho_j(n_j)}\}, \{\sigma_{\rho_j(n_j)}\}\rangle \, . \tag{62}$$

where K is the total number of particle species and $\{p_{n_j}\}$, $\{\sigma_{n_j}\}$ the set of momenta and spin projections of particles of species j in $|f\rangle$. Also let $\boldsymbol{\rho}$ be the set $(\rho_1, \dots, \rho_K)$ of permutations of integers $(1, \dots, N_1), \dots, (1, \dots, N_K)$ and let $N = \sum_{j=1}^{k} N_j$. Let us also introduce a shorthand $|f_{\boldsymbol{\rho}}\rangle$ for the ket:

$$|f_{\boldsymbol{\rho}}\rangle \equiv \prod_{j=1}^{K} |N_j, \{p_{\rho_j(n_j)}\}, \{\sigma_{\rho_j(n_j)}\}\rangle \tag{63}$$

which is the multiparticle tensor formed by exchanging particle indices by the permutations $\boldsymbol{\rho}$. By using this notation, the state $|f\rangle$ in (62) can be rewritten as:

$$|f\rangle = \sum_{\boldsymbol{\rho}} \left[\prod_{j=1}^{k} \frac{\chi(\rho_j)^{b_j}}{\sqrt{N_j!}} \right] |f_{\boldsymbol{\rho}}\rangle \tag{64}$$

and the microcanonical state weight ω_f:

$$\omega_f = \sum_{\boldsymbol{\eta}} \sum_{\boldsymbol{\rho}} \left[\prod_{j=1}^{K} \frac{\chi(\rho_j \eta_j)^{b_j}}{N_j!} \right] \langle f_{\boldsymbol{\eta}}|\mathsf{P}_i \mathsf{P}_V|f_{\boldsymbol{\rho}}\rangle \tag{65}$$

which can be simplified, redefining dummy particle indices, into:

$$\omega_f = \sum_{\boldsymbol{\rho}} \left[\prod_{j=1}^{k} \chi(\rho_j)^{b_j} \right] \langle f_{\boldsymbol{\iota}} | \mathsf{P}_i \mathsf{P}_V | f_{\boldsymbol{\rho}} \rangle \tag{66}$$

where $\boldsymbol{\iota}$ is the identical permutation.

We first consider, for sake of simplicity, a multiparticle state with N identical particles. Eq. (66) reads, by taking $\mathsf{P}_i \equiv \mathsf{P}_{PJ\lambda}$:

$$\begin{aligned} \omega_f =& \sum_{\rho} \chi(\rho)^b \langle f_\iota | \mathsf{P}_{PJ\lambda}\ \mathsf{P}_V | f_\rho \rangle \\ =& \sum_{\rho} \chi(\rho)^b \langle N, \{p_n\}, \{\sigma_n\} | \mathsf{P}_{PJ\lambda}\ \mathsf{P}_V | N, \{p_{\rho(n)}\}, \{\sigma_{\rho(n)}\} \rangle \end{aligned} \tag{67}$$

Like for eq. (19), we insert a resolution of the identity between P_i and P_V leading to:

$$\begin{aligned} \omega_f =& \sum_{\rho} \chi(\rho)^b \left[\prod_{n=1}^{N} \sum_{\sigma'_n} \int \mathrm{d}^3 \mathrm{p}'_n \right] \langle N, \{p_n\}, \{\sigma_n\} | \mathsf{P}_{PJ\lambda} | N, \{p'_n\}, \{\sigma'_n\} \rangle \\ & \times \langle N, \{p'_n\}, \{\sigma'_n\} | \mathsf{P}_V | N, \{p_{\rho(n)}\}, \{\sigma_{\rho(n)}\} \rangle \ . \end{aligned} \tag{68}$$

Both matrix elements of P_V and $\mathsf{P}_{PJ\lambda}$ can be evaluated, in the multiparticle tensor space, the same way as for distinct particles. By using eqs. (57) and (58):

$$\begin{aligned} \omega_f =& \langle 0 | \mathsf{P}_V | 0 \rangle \delta^4 \left(P - \sum_{n=1}^{N} p_n \right) \left[\prod_{n=1}^{N} \sum_{\sigma'_n} \int \mathrm{d}^3 \mathrm{p}'_n \right] (2J+1) \\ & \times \sum_{\rho} \chi(\rho)^b \int \mathrm{dR}\ D^J_{\lambda\lambda}(\mathsf{R}^{-1}) \left[\prod_{n=1}^{N} \delta^3(\mathsf{R}\mathbf{p}'_n - \mathbf{p}_n) D^S_{\lambda_n \lambda'_n}([\mathsf{R}p'_n]^{-1} \mathsf{R}[p'_n]) \right. \\ & \left. \times \sqrt{\frac{\varepsilon_{\rho(n)}}{\varepsilon'_n}} F_V(\mathbf{p}_{\rho(n)} - \mathbf{p}'_n) \frac{1}{2} \left(D^S([p'_n]^{-1}[p_{\rho(n)}]) + D^S([p'_n]^{\dagger}[p_{\rho(n)}]^{\dagger -1}) \right)_{\sigma'_n \sigma_{\rho(n)}} \right] \end{aligned} \tag{69}$$

If we now carry out the integration over momenta p'_i and sum over all spin projections σ'_n we get:

$$
\begin{aligned}
\omega_f = \delta^4 \left(P - \sum_{n=1}^{N} p_n\right) (2J+1) \int \mathrm{dR}\; D^J_{\lambda\lambda}(\mathsf{R}^{-1}) \sum_{\rho} \chi(\rho)^b \\
\times \left[\prod_{n=1}^{N} \frac{1}{2} \left(D^S([p_n]^{-1}\mathsf{R}[p_{\rho(n)}]) + D^S([p_n]^{\dagger}\mathsf{R}[p_{\rho(n)}]^{\dagger -1}) \right)_{\sigma_n \sigma_{\rho(n)}} \right. \\
\times \left. \sqrt{\frac{\varepsilon_{\rho(n)}}{\varepsilon_n}}\, F_V(\mathbf{p}_{\rho(n)} - \mathsf{R}^{-1}\mathbf{p}_n) \right] \langle 0|\mathsf{P}_V|0\rangle \; . \qquad (70)
\end{aligned}
$$

Indeed, the factor $\prod_{n=1}^{N} \sqrt{\varepsilon_{\rho(n)}/\varepsilon_n}$ can be dropped from eq. (70) because numerator and denominator are altogether equal being ρ a permutation.

The microcanonical channel weight of N identical particles can be calculated by integrating ω_f in (70):

$$
\begin{aligned}
\Omega_N = \langle 0|\mathsf{P}_V|0\rangle (2J+1) \int \mathrm{dR}\; D^J_{\lambda\lambda}(\mathsf{R}^{-1}) \sum_{\sigma_1,\ldots,\sigma_N} \sum_{\rho} \frac{\chi(\rho)^b}{N!} \left[\prod_{n=1}^{N} \int \mathrm{d}^3\mathrm{p}_n \right. \\
\times \left. \frac{1}{2} \left(D^S([p_n]^{-1}\mathsf{R}[p_{\rho(n)}]) + D^S([p_n]^{\dagger}\mathsf{R}[p_{\rho(n)}]^{\dagger -1}) \right)_{\sigma_n \sigma_{\rho(n)}} F_V(\mathbf{p}_{\rho(n)} - \mathsf{R}^{-1}\mathbf{p}_n) \right] \\
\times \delta^4 \left(P - \sum_{n=1}^{N} p_n\right) \qquad (71)
\end{aligned}
$$

where the factor $1/N!$ has been introduced in order avoid multiple counting of momenta and spin projections.

The above formula can be further developed in the limit of large volumes, when:

$$
F_V(\mathbf{p}_{\rho(n)} - \mathsf{R}^{-1}\mathbf{p}_n) \underset{V\to\infty}{\longrightarrow} \delta^3(\mathbf{p}_{\rho(n)} - \mathsf{R}^{-1}\mathbf{p}_n)
$$

In this case, the two matrices of the irreducible representation of SL(2,C) in the formula (71) become equal. To show this, one first notes that the Dirac's delta implies $p_{\rho(n)} = \mathsf{R}^{-1} p_n$, hence:

$$
\begin{aligned}
&[p_n]^{-1}\mathsf{R}[p_{\rho(n)}] \to [p_n]^{-1}\mathsf{R}[\mathsf{R}^{-1}p_n] \\
&[p_n]^{\dagger}\mathsf{R}[p_{\rho(n)}]^{\dagger -1} \to [p_n]^{\dagger}\mathsf{R}[\mathsf{R}^{-1}p_n]^{\dagger -1} \qquad (72)
\end{aligned}
$$

The right hand side of the first equation is manifestly a Wigner rotation, hence it is unitary and equal to the right hand side of the second equation. This implies that in the large volume limit:

$$
\frac{1}{2} \left(D^S([p_n]^{-1}\mathsf{R}[p_{\rho(n)}]) + D^S([p_n]^{\dagger}\mathsf{R}[p_{\rho(n)}]^{\dagger -1}) \right) \underset{V\to\infty}{\longrightarrow} D^S([p_n]^{-1}\mathsf{R}[p_{\rho(n)}]) \qquad (73)
$$

We observe that the same equation applies in the non-relativistic limit, where the matrix $[p]$ is unitary, since the associated Lorentz boost in eq. (7) can be approximated with the identity.

An useful development of the formula in the thermodynamic limit can be achieved by decomposing permutations into the product of irreducible cyclic permutations, that is writing:

$$\rho = c_1 \dots c_H \, . \tag{74}$$

Instead of working out the most general case, we will focus on an example which will show more clearly how to take advantage of the above decomposition; the generalization will then be straightforward. Let us then consider a special case, with 5 particles and the permutation:

$$(1,2,3,4,5) \qquad \longrightarrow \qquad (3,1,2,5,4) \tag{75}$$

which can be decomposed in two cyclic permutations:

$$c_1 = (1,2,3) \longrightarrow (3,1,2) \qquad \text{and} \qquad c_2 = (4,5) \longrightarrow (5,4) \, . \tag{76}$$

If we now write down the corresponding matrices product in eq. (71), by using the reduction (73) we have:

$$\begin{aligned}\prod_{n=1}^{5} D^S_{\sigma_n \sigma_{\rho(n)}}([p_n]^{-1}\mathsf{R}[p_{\rho(n)}]) &= D^S_{\sigma_4\sigma_5}([p_4]^{-1}\mathsf{R}[p_5]) D^S_{\sigma_5\sigma_4}([p_5]^{-1}\mathsf{R}[p_4]) \\ &\times D^S_{\sigma_1\sigma_3}([p_1]^{-1}\mathsf{R}[p_3]) D^S_{\sigma_3\sigma_2}([p_3]^{-1}\mathsf{R}[p_2]) D^S_{\sigma_2\sigma_1}([p_2]^{-1}\mathsf{R}[p_1])\end{aligned}$$

where the first two factors on the right hand side correspond to the cyclic permutation c_2, while the other three to c_1. Summing then over spin projections σ_n, we end up with the simple and nice result:

$$\begin{aligned}\sum_{\{\sigma_n\}} \prod_{n=1}^{5} D^S_{\sigma_n \sigma_{\rho(n)}}([p_n]^{-1}\mathsf{R}[p_{\rho(n)}]) &= \mathrm{tr} D^S([p_4]^{-1}\mathsf{R}^2[p_4]) \mathrm{tr} D^S([p_1]^{-1}\mathsf{R}^3[p_1]) \\ &= \mathrm{tr} D^S(\mathsf{R}^2) \mathrm{tr} D^S(\mathsf{R}^3)\end{aligned} \tag{77}$$

In general, for each permutation ρ and its cyclic decomposition, if we let l be the number of integers in each cyclic permutation and h_l the number of cyclic permutations with l elements in ρ such that $\sum_{l=1}^{\infty} l h_l = N$ [3], one has:

$$\sum_{\{\sigma\}} \prod_{n=1}^{N} D^S_{\sigma_n \sigma_{\rho(n)}}([p_n]^{-1}\mathsf{R}[p_{\rho(n)}]) = \prod_{l=1}^{N} \left[\mathrm{tr} D^S(\mathsf{R}^l) \right]^{h_l(\rho)} \tag{78}$$

Thus, by using eq. (73) and its consequence eq. (78) in the large volume limit the microcanonical channel weight (71) turns into:

[3] The set of integers $h_1, \dots, h_N \equiv \{h_l\}$, is usually defined as a *partition* of the integer N in the multiplicity representation.

$$\Omega_N = (2J+1) \int \mathrm{dR}\; D^J_{\lambda\lambda}(\mathsf{R}^{-1}) \sum_{\rho} \frac{\chi(\rho)^b}{N!} \left[\prod_{n=1}^{N} \int \mathrm{d}^3\mathrm{p}_n \right.$$

$$\left. \times\, F_V(\mathbf{p}_{\rho(n)} - \mathsf{R}^{-1}\mathbf{p}_n) \left[\mathrm{tr} D^S(\mathsf{R}^n) \right]^{h_n(\rho)} \right] \delta^4 \left(P - \sum_{n=1}^{N} p_n \right) \langle 0 | \mathsf{P}_V | 0 \rangle \, . \quad (79)$$

an expression which also applies to the non-relativistic limit. On the other hand, if V is not large and we have a full relativistic regime, the two D^S matrices in eq. (71) are no longer equal and the summation over polarizations gives rise to a very involved expression. The general (71) and the special case (79) can be easily extended to a multi-species ideal gas. They read:

$$\Omega_{\{N_j\}} = (2J+1) \int \mathrm{dR}\; D^J_{\lambda\lambda}(\mathsf{R}^{-1}) \sum_{\sigma_1,\ldots,\sigma_N} \sum_{\boldsymbol{\rho}} \left[\prod_{j=1}^{K} \frac{\chi(\rho_j)^{b_j}}{N_j!} \prod_{n_j=1}^{N_j} \int \mathrm{d}^3\mathrm{p}_{n_j} \right.$$

$$\times \frac{1}{2} \left(D^{S_j}([p_{n_j}]^{-1}\mathsf{R}[p_{\rho(n_j)}]) + D^{S_j}([p_{n_j}]^{\dagger}\mathsf{R}[p_{\rho(n_j)}]^{\dagger-1}) \right)_{\sigma_{n_j}\sigma_{\rho(n_j)}}$$

$$\left. \times\, F_V(\mathbf{p}_{\rho(n_j)} - \mathsf{R}^{-1}\mathbf{p}_{n_j}) \right] \delta^4 \left(P - \sum_{n=1}^{N} p_n \right) \langle 0 | \mathsf{P}_V | 0 \rangle \quad (80)$$

and:

$$\Omega_{\{N_j\}} = (2J+1) \int \mathrm{dR}\; D^J_{\lambda\lambda}(\mathsf{R}^{-1}) \sum_{\boldsymbol{\rho}} \left[\prod_{j=1}^{K} \frac{\chi(\rho_j)^{b_j}}{N_j!} \prod_{n_j=1}^{N_j} \int \mathrm{d}^3\mathrm{p}_{n_j} \right.$$

$$\left. \times\, F_V(\mathbf{p}_{\rho(n_j)} - \mathsf{R}^{-1}\mathbf{p}_{n_j}) \left[\mathrm{tr} D^{S_j}(\mathsf{R}^{n_j}) \right]^{h_{n_j}(\rho_j)} \right] \delta^4 \left(P - \sum_{n=1}^{N} p_n \right) \langle 0 | \mathsf{P}_V | 0 \rangle \quad (81)$$

respectively. In the eq. (81) $\sum_{n_j=1}^{N_j} n_j h_j(\rho_j) = N_j$.

Finally, we observe that the eq. (81) could be further developed in the large volume limit so as to obtain a microcanonical channel weight which is re-summable analytically yielding a compact integral expression of the MPF, just like in the non-relativistic case without angular momentum conservation [6].

In the large volume limit expression (81) there appear traces of rotations. Since the trace is the invariant over group conjugacy classes, it turns out that all SU(2) transformations of the same angle ψ around an axis $\hat{\mathbf{n}}$ (which labels the different members of the same conjugacy class) have the same trace. It is therefore convenient to use the axis-angle parametrization for the integration measure over the SU(2) group:

$$
\begin{aligned}
\int \mathrm{dR} &= \frac{1}{16\pi^2}\int_0^{\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \int_0^{4\pi} \mathrm{d}\psi \; 2\sin\theta \sin^2\frac{\psi}{2} \\
&= \frac{1}{16\pi^2}\int \mathrm{d}\Omega_{\hat{\mathbf{n}}} \int_0^{4\pi} \mathrm{d}\psi \; 2\sin^2\frac{\psi}{2}
\end{aligned} \tag{82}
$$

where (θ, ϕ) are the polar and azimuthal angles defining the axis $\hat{\mathbf{n}}$. With this parametrization, the trace of a rotation $\mathsf{R}_{\hat{\mathbf{n}}}(\psi)$ reads [9]:

$$
\mathrm{tr} D^S \left(\mathsf{R}_{\hat{\mathbf{n}}}(\psi)\right) = \frac{\sin[\left(S+\frac{1}{2}\right)\psi]}{\sin\frac{\psi}{2}} \; . \tag{83}
$$

Furthermore, since:

$$
\mathsf{R}_{\hat{\mathbf{n}}}(\psi_1)\mathsf{R}_{\hat{\mathbf{n}}}(\psi_2) = \mathsf{R}_{\hat{\mathbf{n}}}(\psi_1+\psi_2) \Rightarrow \mathsf{R}_{\hat{\mathbf{n}}}(\psi)^l = \mathsf{R}_{\hat{\mathbf{n}}}(l\psi)
$$

the trace in eq. (81) simply becomes:

$$
\mathrm{tr} D^S(\mathsf{R}_{\hat{\mathbf{n}}}(\psi)^n) = \frac{\sin[(S+\frac{1}{2})n\psi]}{\sin(\frac{n\psi}{2})} \; . \tag{84}
$$

6 Spherical region

The eq. (80) is the most general microcanonical weight of a channel $\{N_j\}$ with the enforcement of energy-momentum and angular momentum conservation; eq. (81) is an approximate expression of (80) which applies for very large volumes. Summing (80) or (81) over all possible channels yields the full microcanonical partition function.

If the system region has spherical symmetry, those expressions can be further simplified because the whole SU(2) group integrand in all the equations is independent of the axis $\hat{\mathbf{n}}$ (see end of Sect. 5). The proof of this statement proceeds in two steps: first, one observes that $\Omega_{\{N_j\}}$ as a whole does not depend on the polarization index λ for a spherical region, so that one can rewrite it summing over λ and dividing by $2J+1$. In the second step, one shows that the overall SU(2) integrand does not depend on the axis $\hat{\mathbf{n}}$. In formulae:

$$
\begin{aligned}
\Omega_{\{N_j\}} &= (2J+1)\int \mathrm{dR} \; D^J(\mathsf{R}^{-1})_{\lambda\lambda} I(\mathsf{R}) = \sum_\lambda \int \mathrm{dR} \; D^J(\mathsf{R}^{-1})_{\lambda\lambda} I(\mathsf{R}) \\
&\equiv \int \mathrm{dR} \; \mathrm{tr} D^J(\mathsf{R}^{-1}) I(\mathsf{R})
\end{aligned} \tag{85}
$$

where neither $\mathrm{tr} D^J(\mathsf{R}^{-1})$ nor $I(\mathsf{R})$ depend on the axis polar coordinates.

Let us prove the first part of our statement by writing the projector $\mathsf{P}_{PJ\lambda}$ for a system at rest (i.e. with $\mathbf{P}=0$) as:

$$\mathsf{P}_{PJ\lambda} = |M,J,\lambda\rangle\langle M,J,\lambda| \tag{86}$$

(where M is the mass of the system) and the channel weight in (18) as:

$$\begin{aligned}\Omega_{\{N_j\}} &= \sum_{\{k\}} \langle \{N_j\},\{k\}|M,J,\lambda\rangle\langle M,J,\lambda|\mathsf{P}_V|\{N_j\},\{k\}\rangle \\ &= \langle M,J,\lambda|\mathsf{P}_V \left(\sum_{\{k\}} |\{N_j\},\{k\}\rangle\langle\{N_j\},\{k\}| \right) |M,J,\lambda\rangle \ .\end{aligned} \tag{87}$$

Evidently, for spherically symmetric systems the condition:

$$[\mathsf{P}_V, \hat{\mathsf{R}}] = 0 \tag{88}$$

must hold, because P_V (see e.g. eq. (16)) involves the sum of all possible field configurations within a spherically symmetric region and the representation of a rotation on the Hilbert space transforms P_V into itself. Likewise, the operator:

$$\mathsf{P}_{\{N_j\}} \equiv \sum_{\{p\}} |\{N_j\},\{p\}\rangle\langle\{N_j\},\{p\}| \tag{89}$$

is also invariant under rotations since it is a sum over all possible kinematical configurations of free states. Thus:

$$[\mathsf{P}_{\{N_j\}}, \hat{\mathsf{R}}] = 0 \tag{90}$$

and, because of Wigner-Eckart theorem:

$$\Omega_{\{N_j\}} = \langle M,J,\lambda|\mathsf{P}_V\mathsf{P}_{\{N_j\}}|M,J,\lambda\rangle = \langle M,J||\mathsf{P}_V\mathsf{P}_{\{N_j\}}||M,J\rangle \ , \tag{91}$$

as $\mathsf{P}_V\mathsf{P}_{\{N_j\}}$ commutes with rotation operators according to (88) and (90). Therefore, the channel weight is independent of λ. One can then sum over spin projections λ the matrix element in all expressions of the microcanonical channel weight, and divide by $(2J+1)$, i.e.:

$$\frac{1}{(2J+1)}\sum_\lambda D^J_{\lambda\lambda}(\mathsf{R}_{\hat{\mathbf{n}}}^{-1}(\psi)) = \frac{1}{(2J+1)}\mathrm{tr}D^J(\mathsf{R}_{\hat{\mathbf{n}}}^{-1}(\psi)) = \frac{1}{(2J+1)}\frac{\sin\left(J+\frac{1}{2}\right)\psi}{\sin\frac{\psi}{2}} \tag{92}$$

where we used $\mathrm{tr}[\mathsf{R}_{\hat{\mathbf{n}}}^{-1}(\psi)] = \mathrm{tr}[\mathsf{R}_{\hat{\mathbf{n}}}(-\psi)] = \mathrm{tr}[\mathsf{R}_{\hat{\mathbf{n}}}(\psi)]$. Thereby, the dependence on $\hat{\mathbf{n}}$ owing to the matrix $D^J(\mathsf{R}_{\hat{\mathbf{n}}}(\psi))$ is removed.

The second step of the proof is to show that the function $I(\mathsf{R})$ in the eq. (85) is also independent of the axis polar coordinates. By comparing eq. (85) with the integral expression of $\mathsf{P}_{J\lambda}$, i.e.

$$\mathsf{P}_{J\lambda} = (2J+1)\int \mathrm{d}\mathsf{R} D^J(\mathsf{R}^{-1})_{\lambda\lambda}\hat{\mathsf{R}}$$

and comparing (87) with (85) one has:

$$I(\mathsf{R}) = \sum_{\{k\}} \langle \{N_j\},\{k\}|\delta^4(P-\hat{P})\hat{\mathsf{R}}\mathsf{P}_V|\{N_j\},\{k\}\rangle = \mathrm{tr}[\delta^4(P-\hat{P})\hat{\mathsf{R}}\mathsf{P}_V\mathsf{P}_{\{N_j\}}] \tag{93}$$

using definition (89). Let us now replace in the above equation the rotation R with axis $\hat{\mathbf{n}}$ with that around the rotated axis $\mathsf{O}\hat{\mathbf{n}}$, which is $\mathsf{O}\mathsf{R}\mathsf{O}^{-1}$. The rigthmost term in eq. (93) turns into:

$$\begin{aligned}
&\mathrm{tr}[\delta^4(P-\hat{P})\hat{\mathsf{O}}\hat{\mathsf{R}}\hat{\mathsf{O}}^{-1}\mathsf{P}_V\mathsf{P}_{\{N_j\}}] = \mathrm{tr}[\hat{\mathsf{O}}\delta^4(P-\hat{P})\hat{\mathsf{R}}\mathsf{P}_V\mathsf{P}_{\{N_j\}}\hat{\mathsf{O}}^{-1}] \\
&= \mathrm{tr}[\delta^4(P-\hat{P})\hat{\mathsf{R}}\mathsf{P}_V\mathsf{P}_{\{N_j\}}]
\end{aligned} \tag{94}$$

where we used eqs. (88),(90) and took advantage of the commutation between $\delta^4(P-\hat{P})$ and rotations if $\mathbf{P}=0$. The eq. (94) proves our statement, namely the function $I(\mathsf{R})$ is independent of the rotation axis and so is the whole group integrand function in eq. (85).

In conclusion, for a spherically symmetric region, one can integrate away the solid angle $\Omega_{\hat{\mathbf{n}}}$ and choose an arbitrary rotation axis, e.g. $\hat{\mathbf{k}}$, in all expressions of the microcanonical channel weight $\Omega_{\{N_j\}}$. The integration measure of the SU(2) group (82) in the axis-angle parametrization effectively reduces to:

$$\int \mathrm{d}\mathsf{R} = \frac{1}{16\pi^2}\int \mathrm{d}\Omega_{\hat{\mathbf{n}}}\mathrm{d}\psi\; 2\sin^2\frac{\psi}{2} \to \frac{1}{4\pi}\int_0^{4\pi}\mathrm{d}\psi\; 2\sin^2\frac{\psi}{2}\;. \tag{95}$$

Particularly, the two equations (80) and (81) for a spherical region become:

$$\begin{aligned}
\Omega_{\{N_j\}} = &\frac{1}{2\pi}\int_0^{4\pi}\mathrm{d}\psi\;\sin\frac{\psi}{2}\sin\left(J+\frac{1}{2}\right)\psi\sum_{\sigma_1,\ldots,\sigma_N}\sum_{\boldsymbol{\rho}}\left[\prod_{j=1}^{K}\frac{\chi(\rho_j)^{b_j}}{N_j!}\prod_{n_j=1}^{N_j}\int \mathrm{d}^3\mathrm{p}_{n_j}\right.\\
&\times\frac{1}{2}\left(D^{S_j}([p_{n_j}]^{-1}\mathsf{R}_3(\psi)[p_{\rho(n_j)}]) + D^{S_j}([p_{n_j}]^{\dagger}\mathsf{R}_3(\psi)[p_{\rho(n_j)}]^{\dagger -1})\right)_{\sigma_{n_j}\sigma_{\rho(n_j)}}\\
&\left.\times F_V(\mathbf{p}_{\rho(n_j)} - \mathsf{R}_3(\psi)^{-1}\mathbf{p}_{n_j})\right]\delta^4\left(P-\sum_{n=1}^{N}p_n\right)\langle 0|\mathsf{P}_V|0\rangle
\end{aligned} \tag{96}$$

and:

$$\Omega_{\{N_j\}} = \frac{1}{2\pi}\int_0^{4\pi} \mathrm{d}\psi \; \sin\frac{\psi}{2}\sin\left(J+\frac{1}{2}\right)\psi \sum_{\boldsymbol{\rho}} \left[\prod_{j=1}^{K} \frac{\chi(\rho_j)^{b_j}}{N_j!} \prod_{n_j=1}^{N_j} \int \mathrm{d}^3\mathrm{p}_{n_j} \right.$$
$$\left. \times\, F_V(\mathbf{p}_{\rho(n_j)} - \mathsf{R}_3^{-1}(\psi)\mathbf{p}_{n_j}) \left[\frac{\sin(S_j+\frac{1}{2})n_j\psi}{\sin(\frac{n_j\psi}{2})}\right]^{h_{n_j}(\rho_j)}\right] \delta^4\left(P - \sum_{n=1}^{N} p_n\right) \langle 0|\mathsf{P}_V|0\rangle \tag{97}$$

7 Partial wave expansion

So far, we have been using a plane wave expansion to calculate the microcanonical channel weight. One may wonder whether an equivalent formula could be obtained using single particle states with definite angular momentum. In this section, we will show that this is the case.

Instead of starting from scratch pur calculation by expressing the microcanonical state weight (19) with kets $|jmS\sigma\rangle$ or $|jmlS\rangle$ (in the lS-coupling), we will work out the formula eq. (80) expanding plane waves into partial waves. For the sake of simplicity, we restrict to the simple case of Boltzmann statistics, corresponding to retain only the permutation identity in the sum in (80), yielding:

$$\begin{aligned}
\Omega_{\{N_j\}} =& (2J+1)\int \mathrm{d}\mathsf{R}\; D^J_{\lambda\lambda}(\mathsf{R}^{-1}) \left[\prod_{j=1}^{K}\frac{1}{N_j!}\prod_{n_j=1}^{N_j}\int \mathrm{d}^3\mathrm{p}_{n_j}\right. \\
&\left. \times\, \frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathrm{x}\, \mathrm{e}^{i\mathbf{x}\cdot(\mathbf{p}_{n_j}-\mathsf{R}^{-1}\mathbf{p}_{n_j})}\;\mathrm{tr} D^{S_j}(\mathsf{R})\right]\delta^4\left(P-\sum_{n=1}^{N}p_n\right)\langle 0|\mathsf{P}_V|0\rangle \\
=& (2J+1)\int \mathrm{d}\mathsf{R}\; D^J_{\lambda\lambda}(\mathsf{R}^{-1}) \left[\prod_{j=1}^{K}\frac{1}{N_j!}\right]\left[\prod_{n=1}^{N}\int \mathrm{d}^3\mathrm{p}_n\right. \\
&\left. \times\, \frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathrm{x}\, \mathrm{e}^{i\mathbf{x}\cdot(\mathbf{p}_{n}-\mathsf{R}^{-1}\mathbf{p}_{n})}\;\mathrm{tr} D^{S_n}(\mathsf{R})\right]\delta^4\left(P-\sum_{n=1}^{N}p_n\right)\langle 0|\mathsf{P}_V|0\rangle
\end{aligned} \tag{98}$$

where we used the explicit form of integrals (50) and performed the sum over polarization states in eq. (80) for the simple case of only identical permutation. In the second equality we dropped different indices for different particle species and used, for simplicity, a common index n for particles in the channels.

The exponentials in Fourier integrals in eq. (98) can now be expanded into partial waves according to the well known formulae:

$$
\begin{aligned}
\mathrm{e}^{i\mathbf{x}\cdot\mathbf{p}_n} &= \sum_{l_n=0}^{\infty}\sum_{m_n=-l_n}^{l_n} i^{\,l_n} 4\pi j_{l_n}(\mathrm{p}_n\mathrm{x}) Y_{l_n m_n}(\hat{\mathbf{x}}) Y^*_{l_n m_n}(\hat{\mathbf{p}}_n) \qquad (99)\\
\mathrm{e}^{-i\mathbf{x}\cdot\mathsf{R}\mathbf{p}_n} &= \sum_{l'_n=0}^{\infty}\sum_{m'_n=-l'_n}^{l'_n} (-i)^{\,l'_n} 4\pi j_{l'_n}(\mathrm{p}_n\mathrm{x}) Y^*_{l'_n m'_n}(\hat{\mathbf{x}}) Y_{l'_n m'_n}(\mathsf{R}\hat{\mathbf{p}}_n)
\end{aligned}
$$

where j_l are spherical Bessel functions and Y are spherical harmonics; $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$ are the unit vectors of $\mathbf{x}$ and $\mathbf{p}$ respectively. Consequently, Fourier integrals can be rewritten as:

$$
\begin{aligned}
&\frac{1}{(2\pi)^3}\int_V \mathrm{d}^3\mathrm{x}\ \mathrm{e}^{i\mathrm{x}\cdot(\mathbf{p}_n - \mathsf{R}\mathbf{p}_n)} = (4\pi)^2 \sum_{l_n,m_n;\, l'_n,m'n} i^{\,l_n}(-i)^{\,l'_n} Y^*_{l_n m_n}(\hat{\mathbf{p}}_n) Y_{l'_n m'_n}(\mathsf{R}\hat{\mathbf{p}}_n)\\
&\times \frac{1}{(2\pi)^3}\int_V \mathrm{d}^3\mathrm{x}\ j_{l_n}(\mathrm{p}_n\mathrm{x}) j_{l'_n}(\mathrm{p}_n\mathrm{x}) Y_{l_n m_n}(\hat{\mathbf{x}}) Y^*_{l'_n m'_n}(\hat{\mathbf{x}})\ . \qquad (100)
\end{aligned}
$$

Switching to spherical coordinates $\mathrm{d}^3\mathrm{x} \to \mathrm{x}^2\,\mathrm{dx}\,\mathrm{d}\Omega_{\hat{\mathbf{x}}}$, the angular integration can be made at once yielding:

$$
\int \mathrm{d}\Omega_{\hat{\mathbf{x}}} Y_{lm}(\hat{\mathbf{x}}) Y^*_{l'm'}(\hat{\mathbf{x}}) = \delta_{ll'}\delta_{mm'}\ , \qquad (101)
$$

hence eq. (100) turns into:

$$
\begin{aligned}
&\frac{1}{(2\pi)^3}\int_V \mathrm{d}^3\mathrm{x}\ \mathrm{e}^{i\mathrm{x}\cdot(\mathbf{p}_n - \mathsf{R}\mathbf{p}_n)}\\
&= \frac{(4\pi)^2}{(2\pi)^3}\sum_{l_n,m_n} Y^*_{l_n m_n}(\hat{\mathbf{p}}_n) Y_{l_n m_n}(\mathsf{R}\hat{\mathbf{p}}_n) \int_V \mathrm{dx}\ \mathrm{x}^2 j_l^2(\mathrm{p}_n\mathrm{x})\ . \qquad (102)
\end{aligned}
$$

In eq. (98), for each particle, a factor $\mathrm{tr}D^S(\mathsf{R})$ appears. By taking advantage of the relation:

$$
Y_{lm}(\mathsf{R}\hat{\mathbf{p}}) = \sum_{m'} Y_{lm'}(\hat{\mathbf{p}})\ D^l_{m'm}(\mathsf{R})^* \qquad (103)
$$

the product between the trace and $Y_{lm}(\mathsf{R}\hat{\mathbf{p}}_n)$ can be worked out as:

$$
\begin{aligned}
&\mathrm{tr}D^{S_n}(\mathsf{R}) \sum_{m'_n} Y_{l_n m'_n}(\hat{\mathbf{p}}_n)\ D^{l_n}_{m_n m'_n}(\mathsf{R})\\
&= \sum_{m'_n,\sigma_n} Y_{l_n m'_n}(\hat{\mathbf{p}}_n)\ D^{S_n}_{\sigma_n\sigma_n}(\mathsf{R}) D^{l_n}_{m_n m'_n}(\mathsf{R})\\
&= \sum_{m'_n,\sigma_n} Y_{l_n m'_n}(\hat{\mathbf{p}}_n)\ \langle S_n,\sigma_n; l_n, m'_n|\hat{\mathsf{R}}|S_n,\sigma_n; l_n, m_n\rangle\\
&= \sum_{\substack{J_n,\mu_n,\mu'_n\\ m'_n,\sigma_n}} Y_{l_n m'_n}(\hat{\mathbf{p}}_n)\ \langle\sigma_n, m'_n|S_n\, l_n|J_n,\mu_n\rangle\langle J_n,\mu'_n|S_n\, l_n|\sigma_n, m_n\rangle D^{J_n}_{\mu_n\mu'_n}(\mathsf{R}) \qquad (104)
\end{aligned}
$$

where $\langle m_1, m_2 | j_1, j_2 | j, m \rangle$ is a shorthand for the Clebsch-Gordan coefficient $\langle j_1, m_1; j_2, m_2 | j, m, j_1, j_2 \rangle$. In the last step, the rotation $\hat{\mathsf{R}}$ has been expanded over a complete set of vectors according to:

$$\hat{\mathsf{R}} = \sum_{J,\mu} \sum_{J',\mu'} |J,\mu\rangle\langle J,\mu|\hat{\mathsf{R}}|J',\mu'\rangle\langle J',\mu'| = \sum_{J,\mu} \sum_{J',\mu'} |J,\mu\rangle\langle J',\mu'| \; D^J_{\mu\mu'}(\mathsf{R})\delta_{JJ'} \; . \tag{105}$$

For each particle in the channel, a factor like (104) appears. The SU(2) group integral in eq. (98) can now be calculated collecting all factors depending on R:

$$\begin{aligned}
&(2J+1) \int \mathrm{d}\mathsf{R} \; D^J_{\lambda\lambda}(\mathsf{R}^{-1}) D^{J_1}_{\mu_1\mu'_1}(\mathsf{R}) \ldots D^{J_N}_{\mu_N\mu'_N}(\mathsf{R}) \\
&= (2J+1)\langle J_1, \mu_1; \ldots; J_N, \mu_N| \int \mathrm{d}\mathsf{R} \; D^J_{\lambda\lambda}(\mathsf{R}^{-1}) \, \hat{\mathsf{R}} |J_1, \mu'_1; \ldots; J_N, \mu'_N\rangle \\
&= \langle J_1, \mu_1; \ldots; J_N, \mu_N| \mathsf{P}_{J\lambda} |J_1, \mu'_1; \ldots; J_N, \mu'_N\rangle \\
&= \langle J_1, \mu_1; \ldots; J_N, \mu_N|J,\lambda\rangle\langle J,\lambda|J_1, \mu'_1; \ldots; J_N, \mu'_N\rangle
\end{aligned} \tag{106}$$

Collecting previous results, eqs .(99)-(106), the microcanonical channel weight (98) can be rewritten as:

$$\begin{aligned}
\Omega_{\{N_j\}} &= \left[\prod_{j=1}^{k} \frac{1}{N_j!} \right] \left[\prod_{n=1}^{N} \int \mathrm{d}^3\mathrm{p}_n \right] \delta^4 \left(P - \sum_{n=1}^{N} p_n \right) \\
&\times \sum_{\substack{J_n,\mu_n,\mu'_n \\ l_n, m'_n, \sigma_n, m_n}} \langle J_1, \mu_1; \ldots; J_N, \mu_N|J,\lambda\rangle\langle J,\lambda|J_1, \mu'_1; \ldots; J_N, \mu'_N\rangle \\
&\times \prod_{n=1}^{N} \frac{2}{\pi} \int_V \mathrm{dx}_n \; \mathrm{x}_n^2 \; j^2_{l_n}(\mathrm{p}_n \mathrm{x}_n) Y^*_{l_n m_n}(\hat{\mathbf{p}}_n) Y_{l_n m'_n}(\hat{\mathbf{p}}_n) \langle \sigma_n, m'_n | S_n \, l_n | J_n, \mu_n \rangle \\
&\times \langle J_n, \mu'_n | S_n \, l_n | \sigma_n, m_n \rangle
\end{aligned} \tag{107}$$

This expression finally gives a clear physical meaning to our manipulations: the microcanonical channel weight of the channel is obtained by summing over all possible single-particle states $|p_n, l_n, m_n, S_n, \sigma_n\rangle$, l_n being the orbital angular momentum, S_n the spin angular momentum, m_n and σ_n their respective third components; and projecting these states first on states with total particle angular momentum J_n and then all of them onto a state with total angular momentum J and third component λ.

A considerable simplification occurs if, in eq. (107) momentum conservation constraint is dropped. In this case the δ^4 reduces to $\delta\left(M - \sum_{n=1}^{N} \varepsilon_n\right)$ and the integration over the angular coordinates of three-momenta can be done at once yielding:

$$\int \mathrm{d}\Omega_{\hat{\mathbf{p}}_n} Y^*_{l_n m_n}(\hat{\mathbf{p}}_n) Y_{l_n m'_n}(\hat{\mathbf{p}}_n) = \delta_{m_n m'_n} \; . \tag{108}$$

Thereby, the eq. (107) reduces to, using the unitarity of Clebsch-Gordan coefficients:

$$\begin{aligned}
\Omega_{\{N_j\}} &= \left[\prod_{j=1}^{k}\frac{1}{N_j!}\right]\left[\prod_{n=1}^{N}\int \mathrm{dp}_n\mathrm{p}_n^2\right]\delta\left(M-\sum_{n=1}^{N}\varepsilon_n\right) \\
&\times \sum_{J_n,\mu_n,l_n}\langle J_1,\mu_1;\ldots;J_N,\mu_N|J,\lambda\rangle\langle J,\lambda|J_1,\mu'_1;\ldots;J_N,\mu'_N\rangle \\
&\times \prod_{n=1}^{N}\frac{2}{\pi}\int_V \mathrm{dx}_n\ \mathrm{x}_n^2\ j_{l_n}^2(\mathrm{p}_n\mathrm{x}_n)\ .
\end{aligned} \tag{109}$$

Let us now restore momentum conservation. In the rest frame of the system where:

$$\delta^4\left(P-\sum_{n=1}^{N}p_n\right)=\delta\left(M-\sum_{n=1}^{N}\varepsilon_n\right)\delta^3\left(\sum_{n=1}^{N}\mathbf{p}_n\right)$$

the δ^3 is expand into spherical harmonics:

$$\begin{aligned}
\delta^3\left(\sum_{n=1}^{N}\mathbf{p}_n\right) &= \frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathrm{X}\ \mathrm{e}^{i\sum_{n=1}^{N}\mathbf{p}_n\cdot\mathbf{X}} = \frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathrm{X}\ \prod_{n=1}^{N}\mathrm{e}^{i\mathbf{p}_n\cdot\mathbf{X}} \\
&= \frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathrm{X}\ \prod_{n=1}^{N}\sum_{L_n,k_n} i^{L_n}4\pi\ j_{L_n}(\mathrm{p}_n\mathrm{X})Y^*_{L_n,k_n}(\hat{\mathbf{p}}_n)Y_{L_n,k_n}(\hat{\mathbf{X}})\ .
\end{aligned} \tag{110}$$

where $\hat{\mathbf{X}}$ is the unit vector of $\mathbf{X}$. Plugging eq. (110) into (98), we are left with an integration of the product of three spherical harmonics having the same versor $\hat{\mathbf{p}}_n$ as argument. Switching to polar coordinates:

$$\mathrm{d}^3\mathrm{p}_n \longrightarrow \mathrm{dp}_n\,\mathrm{p}_n^2\mathrm{d}\Omega_{\hat{\mathbf{p}}_n} \qquad \text{and} \qquad \mathrm{d}^3\mathrm{X} \longrightarrow \mathrm{dX}\,\mathrm{X}^2\mathrm{d}\Omega_{\hat{\mathbf{X}}}$$

we have [12]:

$$\begin{aligned}
&\int \mathrm{d}\Omega_{\hat{\mathbf{p}}_n}\ Y^*_{l_n m_n}(\hat{\mathbf{p}}_n)Y_{l_n m'_n}(\hat{\mathbf{p}}_n)Y^*_{L_n,k_n}(\hat{\mathbf{p}}_n) \\
&= \sqrt{\frac{2L_n+1}{4\pi}}\langle 0,0|l_nL_n|l_n,0\rangle\langle m_n,k_n|l_nL_n|l_n,m'_n\rangle\ .
\end{aligned} \tag{111}$$

Furthermore, by recalling the definition of spherical harmonic:

$$\prod_{n=1}^{N}Y_{L_n,k_n}(\hat{\mathbf{X}})=\prod_{n=1}^{N}\sqrt{\frac{2L_n+1}{4\pi}}D^{L_n}_{k_n0}(\varphi,\theta,0)^* \tag{112}$$

where θ and φ are, respectively, the polar and azimuthal angles of the unit vector $\hat{\mathbf{X}}$, we can solve the angular integration in $\Omega_{\hat{\mathbf{X}}}$:

$$
\begin{aligned}
&\int \mathrm{d}\Omega_{\hat{\mathbf{x}}} \prod_{n=1}^{N} \sqrt{\frac{2L_n+1}{4\pi}} D^{L_n}_{k_n 0}(\varphi,\theta,0)^* \\
&= \int \mathrm{d}\Omega_{\hat{\mathbf{x}}} \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\psi \prod_{n=1}^{N} \sqrt{\frac{2L_n+1}{4\pi}} D^{L_n}_{k_n 0}(\varphi,\theta,\psi)^* \\
&= \prod_{n=1}^{N} \sqrt{\frac{2L_n+1}{4\pi}} 4\pi \int \mathrm{dR} \prod_{n=1}^{N} D^{L_n}_{0 k_n}(\mathsf{R}^{-1}) \\
&= \prod_{n=1}^{N} \sqrt{\frac{2L_n+1}{4\pi}} 4\pi \langle L_1, 0; \ldots ; L_N, 0 | 0, 0 \rangle \langle 0, 0 | L_1, k_1; \ldots ; L_N, k_N \rangle
\end{aligned}
\tag{113}
$$

where we have used the unitarity of R and the last equality is a special case of eq. (106).

Finally, the microcanonical channel weight reads:

$$
\begin{aligned}
\Omega_{\{N_j\}} = &\left[\prod_{j=1}^{k} \frac{1}{N_j!} \right] \int_0^{\infty} \mathrm{dX}\ \mathrm{X}^2 \left[\prod_{n=1}^{N} \int \mathrm{dp}_n\ \mathrm{p}_n^2 \right] \delta \left(M - \sum_{n=1}^{N} \varepsilon_n \right) \\
&\times \sum_{\substack{J_n, \mu_n, \mu_n', L_n, k_n \\ l_n, m_n', \sigma_n, m_n}} \left\{ \prod_{n=1}^{N} \frac{2L_n+1}{4\pi} j_{L_n}(\mathrm{p}_n \mathrm{X}) \frac{2}{\pi} \int_V \mathrm{dx}_n\ \mathrm{x}_n^2\ j^2_{l_n}(\mathrm{p}_n \mathrm{x}_n) \langle 0, 0 | l_n L_n | l_n, 0 \rangle \right. \\
&\left. \times \langle m_n, k_n | l_n L_n | l_n, m_n' \rangle \langle \sigma_n, m_n' | S_n\, l_n | J_n, \mu_n \rangle \langle J_n, \mu_n' | S_n\, l_n | \sigma_n, m_n \rangle \right\} \\
&\times \langle J_1, \mu_1; \ldots ; J_N, \mu_N | J, \lambda \rangle \langle J, \lambda | J_1, \mu_1'; \ldots ; J_N, \mu_N' \rangle \\
&\times \langle L_1, 0; \ldots ; L_N, 0 | 0, 0 \rangle \langle 0, 0 | L_1, k_1; \ldots ; L_N, k_N \rangle \ .
\end{aligned}
\tag{114}
$$

In the above expression, only $N+1$ integrations are left instead of $3N$, but many series appear which can make the numerical calculation as cumbersome as in the plane wave expansion (80).

8 Classical limit

We now show that the microcanonical channel weight we have calculated in the quantum case reduces to what is expected for a fully classical system. The classical limit of eq. (98) (we will not consider the quantum statistics case) can be obtained by reintroducing $\hbar$ and then taking the limit $\hbar \to 0$. Let us then rewrite eq. (98) this way, with the invariant SU(2) measure in the axis-angle parametrization:

$$\Omega_{\{N_j\}} = (2J+1)\frac{1}{8\pi^2}\int \mathrm{d}\Omega_{\hat{\mathbf{n}}}\int_0^{4\pi}\mathrm{d}\psi\,\sin^2\frac{\psi}{2}\,D^J_{\lambda\lambda}(\mathsf{R}_{\hat{\mathbf{n}}}^{-1})\left[\prod_{j=1}^{K}\frac{1}{N_j!}\right]$$
$$\times\left[\prod_{n=1}^{N}\frac{\mathrm{tr}D^{S_n}(\mathsf{R}_{\hat{\mathbf{n}}}(\psi))}{(2\pi\hbar)^3}\int \mathrm{d}^3\mathrm{p}_n\int_V \mathrm{d}^3\mathrm{x}_n\;\,\mathrm{e}^{i\mathbf{x}_n/\hbar\cdot(\mathbf{p}_n-\mathsf{R}_{\hat{\mathbf{n}}}^{-1}(\psi)\mathbf{p}_n)}\right]\delta^4\left(P-\sum_{n=1}^{N}p_n\right). \tag{115}$$

Setting:

$$\mathbf{y}\equiv\frac{\mathbf{x}}{\hbar}$$

the eq. (115) can be rewritten as:

$$\Omega_{\{N_j\}} = (2J+1)\frac{1}{8\pi^2}\int \mathrm{d}\Omega_{\hat{\mathbf{n}}}\int_0^{4\pi}\mathrm{d}\psi\;\sin^2\frac{\psi}{2}\,D^J_{\lambda\lambda}(\mathsf{R}_{\hat{\mathbf{n}}}(\psi)^{-1})\left[\prod_{j=1}^{K}\frac{1}{N_j!}\right]$$
$$\times\left[\prod_{n=1}^{N}\frac{\mathrm{tr}D^{S_n}(\mathsf{R}_{\hat{\mathbf{n}}}(\psi))}{(2\pi)^3}\int \mathrm{d}^3\mathrm{p}_n\int_{V/\hbar^3} \mathrm{d}^3\mathrm{y}_n\;\,\mathrm{e}^{i\mathbf{y}_n\cdot(\mathbf{p}_n-\mathsf{R}_{\hat{\mathbf{n}}}^{-1}(\psi)\mathbf{p}_n)}\right]\delta^4\left(P-\sum_{n=1}^{N}p_n\right). \tag{116}$$

In the limit $\hbar\to 0$, the integration domain $V/\hbar^3$ of the Fourier integrals in $\mathbf{y}$ becomes very large and the integrals tend to a Dirac's delta distribution. Hence ($\mathbf{p}_n \to \mathsf{R}_{\hat{\mathbf{n}}}^{-1}(\psi)\mathbf{p}_n$) for each $\mathbf{p}_n$ and this means that the rotation $\mathsf{R}_{\hat{\mathbf{n}}}^{-1}$ tends to the identity, or, equivalently $\psi\to 0, 2\pi, 4\pi$. Indeed, it can be shown that for increasingly small $\hbar$, the integrand in eq. (116) develops 4 symmetric narrow gaussian peaks in ψ with maxima at $\psi = \epsilon, 2\pi-\epsilon, 2\pi+\epsilon, 4\pi-\epsilon$ with $\epsilon\to 0$ as $\hbar\to 0$. Hence, we can reduce the integration on the angle ψ to the interval $[0,\pi]$ multiplying (116) by 4 and in this interval approximate $\sin\psi\simeq\psi$ and $\cos\psi\simeq 1$. If $\mathbf{v}$ is a generic vector, the vector $\mathsf{R}_{\hat{\mathbf{n}}}(\psi)\mathbf{v}$ reads:

$$\mathsf{R}_{\hat{\mathbf{n}}}(\psi)\mathbf{v} = \mathbf{v}\cos\psi + (\hat{\mathbf{n}}\times\mathbf{v})\sin\psi + (1-\cos\psi)\mathbf{v}\cdot\hat{\mathbf{n}}\,\hat{\mathbf{n}} \tag{117}$$

therefore, for small ψ:

$$\mathsf{R}_{\hat{\mathbf{n}}}(\psi)\simeq\mathbf{v}+\psi(\hat{\mathbf{n}}\times\mathbf{v}) \tag{118}$$

thus:

$$\mathbf{p}_n-\mathsf{R}_{\hat{\mathbf{n}}}^{-1}(\psi)\mathbf{p}_n\simeq\psi(\hat{\mathbf{n}}\times\mathbf{p}) \tag{119}$$

By using (119) and $\sin\psi\simeq\psi$ in (116) we obtain:

$$\Omega_{\{N_j\}} \underset{\hbar\to 0}{\simeq} (2J+1)\frac{4}{8\pi^2}\int \mathrm{d}\Omega_{\hat{\mathbf{n}}}\int_0^{\pi}\mathrm{d}\psi\;\frac{\psi^2}{4}\,D^J_{\lambda\lambda}(\mathsf{R}_{\hat{\mathbf{n}}}^{-1})\left[\prod_{j=1}^{K}\frac{1}{N_j!}\right]$$
$$\times\left[\prod_{n=1}^{N}\frac{1}{(2\pi)^3}\,\mathrm{tr}D^{S_n}(\mathsf{R}_{\hat{\mathbf{n}}}(\psi))\int \mathrm{d}^3\mathrm{p}_n\int_{V/\hbar^3} \mathrm{d}^3\mathrm{y}_n\;\,\mathrm{e}^{i\mathbf{y}_n\cdot\psi(\hat{\mathbf{n}}\times\mathbf{p})}\right]\delta^4\left(P-\sum_{n=1}^{N}p_n\right)\;. \tag{120}$$

Now $\mathrm{tr}D^{S_n}(\mathsf{R}_{\hat{\mathbf{n}}})$ can be replaced with $(2S_n+1)$ if $\psi\to 0$. On the other hand, if $J/\hbar$ and $\lambda/\hbar$ are macroscopic integer numbers, we can also take the large spin limit of the rotation matrix and replace $D^J_{\lambda\lambda}(\mathsf{R}_{\hat{\mathbf{n}}}(\psi)^{-1})$ with $\exp[i\psi\hat{\mathbf{n}}\cdot\mathbf{J}/\hbar]$ where $\mathbf{J}=\lambda\hat{\mathbf{k}}$ is defined as the total angular momentum vector. Therefore:

$$\Omega_{\{N_j\}} \underset{\hbar\to 0}{\simeq} (2J+1)\frac{1}{8\pi^2}\int \mathrm{d}\Omega_{\hat{\mathbf{n}}} \int_0^\pi \mathrm{d}\psi\ \psi^2\ \mathrm{e}^{i\psi\hat{\mathbf{n}}\cdot\mathbf{J}/\hbar}\left[\prod_{j=1}^K \frac{2S_j+1}{N_j!}\right]$$
$$\times\left[\prod_{n=1}^N \frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathrm{p}_n \int_{V/\hbar^3} \mathrm{d}^3\mathrm{y}_n\ \mathrm{e}^{i\mathbf{y}_n\cdot\psi(\hat{\mathbf{n}}\times\mathbf{p})}\right]\delta^4\left(P-\sum_{n=1}^N p_n\right)\ . \tag{121}$$

We can introduce a classical orbital momentum $\mathbf{L}$ by noting that:

$$\mathbf{y}_{n_j}\cdot(\hat{\mathbf{n}}\times\mathbf{p}_{n_j}) = -\hat{\mathbf{n}}\cdot(\mathbf{y}_{n_j}\times\mathbf{p}_{n_j}) = -\hat{\mathbf{n}}\cdot\mathbf{L}_{n_j}/\hbar \tag{122}$$

Also, one can introduce a the three-vector $\boldsymbol{\phi}=\psi\hat{\mathbf{n}}$ and rewriting the eq. (121) with $\mathrm{d}^3\phi\equiv\mathrm{d}\Omega_{\hat{\mathbf{n}}}\mathrm{d}\psi\psi^2$:

$$\Omega_{\{N_j\}} \simeq (2J+1)\frac{1}{8\pi^2}\int \mathrm{d}^3\phi\ \mathrm{e}^{i\boldsymbol{\phi}\cdot\mathbf{J}/\hbar}\left[\prod_{j=1}^K \frac{(2S_j+1)^{N_j}}{N_j!}\right]\left[\prod_{n=1}^N \int \mathrm{d}^3\mathrm{p}_n\right.$$
$$\left.\times\frac{1}{(2\pi)^3}\int_{V/\hbar^3}\mathrm{d}^3\mathrm{y}_n\ \mathrm{e}^{-i\boldsymbol{\phi}\cdot\mathbf{L}_n/\hbar}\right]\delta^4\left(P-\sum_{n=1}^N p_n\right) \tag{123}$$

where the integration in $\boldsymbol{\phi}$ is performed over the whole space as the contribution of the ψ interval $[\pi,+\infty)$ is negligible for $\hbar\to 0$. Now, by restoring the variable $\mathbf{x}=\hbar\mathbf{y}$ and rescaling $\boldsymbol{\phi}\to\boldsymbol{\phi}/\hbar$ we finally obtain:

$$\Omega^{\{N_j\}}_{\mathrm{classical}} = (2J+1)\frac{\pi}{\hbar^{3N-3}}\left[\prod_{j=1}^K \frac{(2S_j+1)^{N_j}}{N_j!}\right]$$
$$\times\left[\prod_{n=1}^N \frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathrm{p}_n \int_V \mathrm{d}^3\mathrm{x}_n\right]\ \delta^4\left(P-\sum_{n=1}^N p_n\right)\delta^3\left(\mathbf{J}-\sum_{n=1}^N \mathbf{L}_n\right)\ . \tag{124}$$

which is the classical expression of the microcanonical channel weight with angular momentum conservation [5]. As expected, in the classical case, this conservation implies a simple δ^3 as though the angular momentum components were commuting observables. To be noted that the contribution of spin reduces to an overall degeneracy factor.

9 Grand-canonical limit

It is also interesting to derive the (grand-)canonical partition function with angular momentum conservation as a limiting case of the microcanonical partition function for large volumes and energies. Again, we will confine ourselves to the case of Boltzmann statistics. One can start from the expression (98) of the microcanonical channel weight and disregard the factor $\langle 0|\mathsf{P}_V|0\rangle$ as $\mathsf{P}_V \to \mathsf{I}$:

$$\Omega_{\{N_j\}} = (2J+1)\frac{1}{8\pi^2}\int \mathrm{d}\Omega_{\hat{\mathbf{n}}} \int_0^{4\pi} \mathrm{d}\psi \; \sin^2\frac{\psi}{2}\, D^J_{\lambda\lambda}(\mathsf{R}_{\hat{\mathbf{n}}}(\psi)^{-1}) \left[\prod_{j=1}^K \frac{1}{N_j!}\right]$$
$$\times \left[\prod_{n=1}^N \int \mathrm{d}^3\mathrm{p}_n \frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathrm{x}\, \mathrm{e}^{i\mathbf{x}\cdot(\mathbf{p}_n - \mathsf{R}_{\hat{\mathbf{n}}}(\psi)^{-1}\mathbf{p}_n)}\; \mathrm{tr} D^{S_n}(\mathsf{R}_{\hat{\mathbf{n}}}(\psi))\right] \delta^4\left(P - \sum_{n=1}^N p_n\right) \tag{125}$$

where we have used the axis-angle parametrization. We have seen in the previous section, dealing with the classical limit, that for large volumes, the integrand in ψ in the above equation develops 4 symmetric gaussian peaks in ψ with maxima at $\psi = \epsilon, 2\pi-\epsilon, 2\pi+\epsilon, 4\pi-\epsilon$ with $\epsilon \to 0$ as $V \to \infty$ and that the Fourier integrals be approximated by:

$$F_V(\mathbf{p}_n - \mathsf{R}_{\hat{\mathbf{n}}}(\psi)^{-1}\mathbf{p}_n) \underset{\psi\to 0}{\simeq} \frac{1}{(2\pi)^3}\int_V \mathrm{d}^3\mathrm{x}\;\, \mathrm{e}^{i\mathbf{x}\cdot\psi(\hat{\mathbf{n}}\times\mathbf{p}_n)} \tag{126}$$

In view of eqs. (117),(118). Thus, plugging (126), into eq. (125) and using the same approximations on the group integration as in Sect. 8:

$$\Omega_{\{N_j\}} = (2J+1)\frac{1}{8\pi^2}\int \mathrm{d}\Omega_{\hat{\mathbf{n}}} \int_0^{\pi} \mathrm{d}\psi \; \psi^2\, D^J_{\lambda\lambda}(\mathsf{R}_{\hat{\mathbf{n}}}(\psi)^{-1}) \left[\prod_{j=1}^K \frac{1}{N_j!}\right]$$
$$\times \left[\prod_{n=1}^N \int_V \mathrm{d}^3\mathrm{p}_n \frac{1}{(2\pi)^3}\int_V \mathrm{d}^3\mathrm{x}_n\, \mathrm{e}^{-i\psi\hat{\mathbf{n}}\cdot(\mathbf{x}_n\times\mathbf{p}_n)}\; \mathrm{tr} D^{S_n}(\mathsf{R}_{\hat{\mathbf{n}}}(\psi))\right] \delta^4\left(P - \sum_{n=1}^N p_n\right) \tag{127}$$

A quick way of deriving the grand-canonical limit of the expression (127) is to replace the δ factor of four momentum conservation with the canonical weight $\prod_{n=1}^N \mathrm{e}^{-\varepsilon_n/T} = \prod_{j=1}^K \prod_{n_j=1}^{N_j} \mathrm{e}^{-\varepsilon_{n_j}/T}$ where T is the temperature. In fact, this procedure corresponds to replace the projector $\delta^4(P-\hat{P})$, with the operator $\exp(-\hat{H}/T)$ so as to match the definition of grand-canonical partition function (with vanishing chemical potentials) with fixed angular momentum:

$$Z_J = \mathrm{tr}[\exp(-\hat{H}/T) P_{J\lambda} \mathsf{P}_V] \tag{128}$$

Indeed, it can be seen from eqs. (15),(21) that (128) is obtained from the microcanonical partition function definition replacing the projector onto fixed energy-momentum with the familiar $\exp[-\hat{H}/T]$. We could have also developed an integral expression of the microcanonical partition function summing up all channels (127) and inserting a Fourier expansion of the δ^4 in order to calculate the grand-canonical limit through a saddle-point expansion [6], but this would imply lengthy calculations. So, with the above replacement, we obtain a grand-canonical channel weight:

$$
\begin{aligned}
Z_{\{N_j\}} &= (2J+1)\frac{1}{8\pi^2}\int \mathrm{d}\Omega_{\hat{\mathbf{n}}} \int_0^\pi \mathrm{d}\psi\ \psi^2\ D^J_{\lambda\lambda}(\mathsf{R}_{\hat{\mathbf{n}}}(\psi)^{-1}) \\
&\times \left[\prod_{j=1}^K \frac{1}{N_j!} \prod_{n_j=1}^{N_j} \int \mathrm{d}^3\mathrm{p}_{n_j} \frac{1}{(2\pi)^3} \int_V \mathrm{d}^3\mathrm{x}_{n_j}\ \mathrm{e}^{-i\psi\hat{\mathbf{n}}\cdot(\mathbf{x}_{n_j}\times\mathbf{p}_{n_j})}\ \mathrm{tr}D^{S_j}(\mathsf{R}_{\hat{\mathbf{n}}}(\psi))\ \mathrm{e}^{-\varepsilon_{n_j}/T} \right] \\
&= (2J+1)\frac{1}{8\pi^2}\int \mathrm{d}\Omega_{\hat{\mathbf{n}}} \int_0^\pi \mathrm{d}\psi\ \psi^2\ D^J_{\lambda\lambda}(\mathsf{R}_{\hat{\mathbf{n}}}(\psi)^{-1}) \\
&\times \left[\prod_{j=1}^K \frac{1}{(2\pi)^3} \int_V \mathrm{d}^3\mathrm{x} \int \mathrm{d}^3\mathrm{p}\ \mathrm{e}^{-\varepsilon_j/T}\ \mathrm{e}^{-i\psi\hat{\mathbf{n}}\cdot(\mathbf{x}\times\mathbf{p})}\ \mathrm{tr}D^{S_j}(\mathsf{R}_{\hat{\mathbf{n}}}(\psi)) \right]^{N_j} .
\end{aligned}
\tag{129}
$$

It is now straightforward to calculate the full grand-canonical partition function by summing over all possible channels, i.e. over all multiplicities $\{N_j\}$ from 0 to ∞. The grand-canonical partition function Z_J with fixed angular momentum turns out to be an SU(2) integral of an exponential function:

$$
\begin{aligned}
Z_J &= \sum_{N_1=0}^{\infty} \ldots \sum_{N_K=0}^{\infty} Z_{\{N_j\}} \\
&= (2J+1)\frac{1}{8\pi^2}\int \mathrm{d}\Omega_{\hat{\mathbf{n}}} \int_0^\pi \mathrm{d}\psi\ \psi^2\ D^J_{\lambda\lambda}(\mathsf{R}_{\hat{\mathbf{n}}}(\psi)^{-1}) \\
&\times \exp\left[\sum_{j=1}^K \frac{1}{(2\pi)^3} \int_V \mathrm{d}^3\mathrm{x} \int \mathrm{d}^3\mathrm{p}\ \mathrm{e}^{-\varepsilon_j/T}\ \mathrm{e}^{-i\psi\hat{\mathbf{n}}\cdot(\mathbf{x}\times\mathbf{p})}\ \mathrm{tr}D^{S_j}(\mathsf{R}_{\hat{\mathbf{n}}}(\psi)) \right]
\end{aligned}
\tag{130}
$$

Defining the vector $\boldsymbol{\phi} = \psi\hat{\mathbf{n}}$ (so that $\hat{\boldsymbol{\phi}} \equiv \hat{\mathbf{n}}$, the eq. (130) can be rewritten as:

$$
\begin{aligned}
Z_J &= (2J+1)\frac{1}{8\pi^2}\int_{|\boldsymbol{\phi}|<\pi} \mathrm{d}^3\boldsymbol{\phi}\ D^J_{\lambda\lambda}(\mathsf{R}_{\hat{\boldsymbol{\phi}}}(\phi)^{-1}) \\
&\times \exp\left[\sum_{j=1}^K \frac{1}{(2\pi)^3} \int_V \mathrm{d}^3\mathrm{x} \int \mathrm{d}^3\mathrm{p}\ \mathrm{e}^{-\varepsilon_j/T}\ \mathrm{e}^{-i\boldsymbol{\phi}\cdot(\mathbf{x}\times\mathbf{p})}\ \mathrm{tr}D^{S_j}(\mathsf{R}_{\hat{\boldsymbol{\phi}}}(\phi)) \right]
\end{aligned}
\tag{131}
$$

For large values of J and λ, we can replace the matrix element in (131) with $\exp[i\boldsymbol{\phi}\cdot\mathbf{J}]$ where $\mathbf{J} = \lambda\hat{\mathbf{k}}$, like in Sect. 8 and write Z_J as:

$$
\begin{aligned}
Z_J =& (2J+1)\frac{1}{8\pi^2}\int_{|\boldsymbol{\phi}|<\pi} \mathrm{d}^3\phi \;\; \mathrm{e}^{i\boldsymbol{\phi}\cdot\mathbf{J}} \\
&\times \exp\left[\sum_{j=1}^{K} \frac{1}{(2\pi)^3}\int_V \mathrm{d}^3\mathrm{x} \int \mathrm{d}^3\mathrm{p} \;\; \mathrm{e}^{-\varepsilon_j/T}\, \mathrm{e}^{-i\boldsymbol{\phi}\cdot(\mathbf{x}\times\mathbf{p})} \;\; \mathrm{tr} D^{S_j}(\mathsf{R}_{\hat{\boldsymbol{\phi}}}(\phi))\right]
\end{aligned}
\qquad (132)
$$

which corresponds to a classical limit were not for the presence of the traces of matrices $D^{S_j}(\mathsf{R}_{\hat{\boldsymbol{\phi}}}(\phi))$.

10 Microcanonical ensemble with fixed parity

In this section we will show how to enforce a fixed parity for an ideal relativistic quantum gas.

As has been shown in Sect. 2, the projector P_i onto irreducible Poincaré states can be factorized as in eq. (14):

$$
\mathsf{P}_i = \mathsf{P}_{PJ\lambda}\frac{\mathsf{I} + \Pi\hat{\mathsf{\Pi}}}{2} \qquad (133)
$$

where Π is the parity of the system. Rewriting the general state weight with the full projector:

$$
\omega_f \equiv \langle f|\mathsf{P}_{PJ\lambda}\frac{\mathsf{I}+\Pi\hat{\mathsf{\Pi}}}{2}\mathsf{P}_V|f\rangle = \frac{1}{2}\langle f|\mathsf{P}_{PJ\lambda}\mathsf{P}_V|f\rangle + \frac{\Pi}{2}\langle f|\mathsf{P}_{PJ\lambda}\hat{\mathsf{\Pi}}\mathsf{P}_V|f\rangle \qquad (134)
$$

we are left with the calculation of two terms: the first is simply eq. (70) times a factor 1/2, while the second is a new one to be calculated. To do that, one should find the action of $\mathsf{\Pi}$ on a single particle state. Indeed, there is no unique answer to this question, as any definition of the unitary representation of space inversion on Hilbert space fulfilling commutation rules of IO(1,3)$^{\uparrow}$ is equally valid. This leaves some freedom and we choose:

$$
\hat{\mathsf{\Pi}}|p,\sigma\rangle = \sum_{\mu} |\mathsf{\Pi}p,\mu\rangle D^S_{\mu\sigma}([\mathsf{\Pi}p]^{-1}\mathsf{\Pi}[p]) \; . \qquad (135)
$$

where $\mathsf{\Pi}p = (\varepsilon, -\mathbf{p})$. To be a good one, the definition (135) must meet the following requirements:

- $D^S([\mathsf{\Pi}p]^{-1}\mathsf{\Pi}[p])$ must be unitary;
- for each state $|p^0,\sigma\rangle$, where $p^0 = (m,\mathbf{0})$, the space inversion $\hat{\mathsf{\Pi}}$ must yield $\hat{\mathsf{\Pi}}|p^0,\sigma\rangle = |p^0,\sigma\rangle\eta$ where η is the intrinsic parity.

With regard to the first point, it should be noted that D^S is a finite dimensional representation of the universal covering group of SO(1,3)$^{\uparrow}_{+}$, i.e. SL(2,C) and a unitary representation of the subgroups SO(3) and SU(2) respectively. For it to be extended so as to include space inversion operator, one should recall the second Shur's lemma:

In a finite-dimensional irreducible representation of a group G, the only elements which commute with all others are multiples of the identity.

Since space inversion commutes with all rotations, i.e. $[\mathsf{\Pi}, \mathsf{R}] = 0$, then:

$$D^S(\mathsf{\Pi})D^S(\mathsf{R}) = D^S(\mathsf{\Pi R}) = D^S(\mathsf{R\Pi}) = D^S(\mathsf{R})D^S(\mathsf{\Pi})$$

thus $D^S(\mathsf{\Pi})$ commutes with all elements of a finite-dimensional representation of a group and, according to the second Schur's lemma, must be a multiple of the identity:

$$D^S(\mathsf{\Pi}) = \eta\, \mathrm{I} \tag{136}$$

Moreover, since $D^S(\mathsf{\Pi}^2) = D^S(\mathsf{I}) = \eta^2\, I$, then $\eta^2 = 1 \Rightarrow \eta = \pm 1$. In the second step of the proof, we show that $[\mathsf{\Pi}p]^{-1}\mathsf{\Pi}[p]$ does not involve any boost, hence it belongs to the subgroup O(3). This can be done by noting that, when acting upon the unit time vector $t = (1, \mathbf{0})$:

$$[\mathsf{\Pi}p]^{-1}\mathsf{\Pi}[p]t \longrightarrow t \qquad \Rightarrow [\mathsf{\Pi}p]^{-1}\mathsf{\Pi}[p] \in \mathrm{O}(3)\ . \tag{137}$$

Since the above transformation belongs to O(3), it is either a rotation or the product of a rotation by $\mathsf{\Pi}$ itself. In both cases, because of (136) and $\eta^2 = 1$, the matrix $D^S([\mathsf{\Pi}p]^{-1}\mathsf{\Pi}[p])$ is unitary and the first requirement is met.

As far as the second requirement is concerned, noting that $[p^0] = [\mathsf{\Pi}p^0] = \mathsf{I}$ and using (135),(136):

$$\begin{aligned} \hat{\mathsf{\Pi}}|p^0, \sigma\rangle &= \sum_\mu |\mathsf{\Pi}p^0, \mu\rangle D^S_{\mu\sigma}([\mathsf{\Pi}p^0]^{-1}\mathsf{\Pi}[p^0]) = \sum_\mu |p^0, \mu\rangle D^S_{\mu\sigma}(\mathsf{\Pi}) \\ &= \sum_\mu |p^0, \mu\rangle \eta \delta_{\mu\sigma} = |p^0, \sigma\rangle\eta \end{aligned} \tag{138}$$

where η is the intrinsic parity. Hence, the second requirement is also met.

We now have to check the consistency of (135) with commutation rules:

$$\begin{aligned} &\hat{\mathsf{\Pi}}\hat{\mathsf{R}}|p, \sigma\rangle = \hat{\mathsf{R}}\hat{\mathsf{\Pi}}|p, \sigma\rangle \\ &\hat{\mathsf{\Pi}}\hat{\mathsf{L}}|p, \sigma\rangle = \hat{\mathsf{L}}^{-1}\hat{\mathsf{\Pi}}|p, \sigma\rangle \qquad\qquad \forall |p, \sigma\rangle\ . \\ &\hat{\mathsf{\Pi}}\hat{\mathbf{P}}|p, \sigma\rangle = -\hat{\mathbf{P}}\hat{\mathsf{\Pi}}|p, \sigma\rangle \end{aligned} \tag{139}$$

R being a rotation, L a pure Lorentz boost and $\hat{\mathbf{P}}$ being the momentum operator. The last equality is trivially fulfilled. For the first commutation rule, going back to definition (135):

$$\begin{aligned}\hat{\mathsf{\Pi}}\hat{\mathsf{R}}|p,\sigma\rangle &= \sum_\tau \hat{\mathsf{\Pi}}|\mathsf{R}p,\tau\rangle D^S_{\tau\sigma}([\mathsf{R}p]^{-1}\mathsf{R}[p]) \\ &= \sum_{\mu,\tau} |\mathsf{\Pi}\mathsf{R}p,\mu\rangle D^S_{\mu\tau}([\mathsf{\Pi}\mathsf{R}p]^{-1}\mathsf{\Pi}[\mathsf{R}p]) D^S_{\tau\sigma}([\mathsf{R}p]^{-1}\mathsf{R}[p]) \\ &= \sum_{\mu} |\mathsf{\Pi}\mathsf{R}p,\mu\rangle D^S_{\mu\sigma}([\mathsf{\Pi}\mathsf{R}p]^{-1}\mathsf{\Pi}\mathsf{R}[p]) \end{aligned} \tag{140}$$

This relation tells us that the composition of operators transforming a state with a Wigner-rotation's rule (like space inversion itself according to (135)) yields an operator fulfilling the same rule. Hence we have:

$$\hat{\mathsf{R}}\hat{\mathsf{\Pi}}|p,\sigma\rangle = \sum_\mu |\mathsf{R}\mathsf{\Pi}p,\mu\rangle D^S_{\mu\sigma}([\mathsf{R}\mathsf{\Pi}p]^{-1}\mathsf{R}\mathsf{\Pi}[p]) \tag{141}$$

and taking into account that $[\mathsf{\Pi},\mathsf{R}]=0$ it follows from eqs. (140) and (141) that $\hat{\mathsf{\Pi}}\hat{\mathsf{R}}|p,\sigma\rangle = \hat{\mathsf{R}}\hat{\mathsf{\Pi}}|p,\sigma\rangle\ \forall|p,\sigma\rangle$.

Similarly, for Lorentz boosts:

$$\begin{aligned}\hat{\mathsf{\Pi}}\hat{\mathsf{L}}|p,\sigma\rangle &= \sum_\mu |\mathsf{\Pi}\mathsf{L}p,\mu\rangle D^S_{\mu\sigma}([\mathsf{\Pi}\mathsf{L}p]^{-1}\mathsf{\Pi}\mathsf{L}[p]) \\ \hat{\mathsf{L}}^{-1}\hat{\mathsf{\Pi}}|p,\sigma\rangle &= \sum_\mu |\mathsf{L}^{-1}\mathsf{\Pi}p,\mu\rangle D^S_{\mu\sigma}([\mathsf{L}^{-1}\mathsf{\Pi}p]^{-1}\mathsf{L}^{-1}\mathsf{\Pi}[p]) \end{aligned} \tag{142}$$

thus, taking into account that $\mathsf{\Pi}\mathsf{L}p = \mathsf{L}^{-1}\mathsf{\Pi}p$ also the last equality in (139) is proved.

The conclusion is that the (135) is consistent with all properties needed by a good definition of space inversion representation.

We can now turn to calculate the second term in eq. (134). We will consider the special case of a spherical system, so that $[\hat{\mathsf{\Pi}},\mathsf{P}_V]=0$, and considering the simple case of a single particle state:

$$\langle p,\sigma|\mathsf{P}_{PJ\lambda}\hat{\mathsf{\Pi}}\mathsf{P}_V|p,\sigma\rangle \tag{143}$$

Since $[\hat{\mathsf{\Pi}},\mathsf{P}_V]=0$, we can rewrite (143) as:

$$\sum_\tau \langle p,\sigma|\mathsf{P}_{PJ\lambda}\mathsf{P}_V|\mathsf{\Pi}p,\tau\rangle D^S_{\tau\sigma}([\mathsf{\Pi}p]^{-1}\mathsf{\Pi}[p]) \tag{144}$$

Inserting a resolution of the identity:

$$\sum_{\tau,\nu}\int \mathrm{d^3p'}\langle p,\sigma|\mathsf{P}_{PJ\lambda}|p',\nu\rangle\langle p',\nu|\mathsf{P}_V|\mathsf{\Pi}p,\tau\rangle D^S_{\tau\sigma}([\mathsf{\Pi}p]^{-1}\mathsf{\Pi}[p])\ . \tag{145}$$

and, by using (28) and (49) we can write:

$$
\langle p,\sigma|\mathsf{P}_{PJ\lambda}\hat{\Pi}\mathsf{P}_V|p,\sigma\rangle = \delta^4(P-p)\int \mathrm{d}^3\mathrm{p}'\ (2J+1)\int \mathrm{dR}\ D^J_{\lambda\lambda}(\mathsf{R}^{-1})\delta^3(\mathsf{R}\mathbf{p}'-\mathbf{p})
$$
$$
\times \sum_{\nu,\tau} D^S_{\sigma\nu}([\mathsf{R}p']^{-1}\mathsf{R}[p'])\frac{1}{2}\left(D^S_{\nu\tau}([p']^{-1}[\Pi p]) + D^S_{\nu\tau}([p']^\dagger[\Pi p]^{\dagger-1})\right) D^S_{\tau\sigma}([\Pi p]^{-1}\Pi[p])
$$
$$
\times F_V(\mathbf{p}+\mathbf{p}')\langle 0|\mathsf{P}_V|0\rangle \tag{146}
$$

The matrix products in (146) can be worked out as follows:

$$
\begin{aligned}
&\sum_{\nu,\tau} D^S_{\sigma\nu}([\mathsf{R}p']^{-1}\mathsf{R}[p'])D^S_{\nu\tau}([p']^{-1}[\Pi p])D^S_{\tau\sigma}([\Pi p]^{-1}\Pi[p]) = D^S_{\sigma\sigma}([p]^{-1}\mathsf{R}\Pi[p])\\
&\sum_{\nu,\tau} D^S_{\sigma\nu}([\mathsf{R}p']^{-1}\mathsf{R}[p'])D^S_{\nu\tau}([p']^\dagger[\Pi p]^{\dagger-1})D^S_{\tau\sigma}([\Pi p]^{-1}\Pi[p])\\
=&\sum_{\nu,\tau} D^S_{\sigma\nu}([\mathsf{R}p']^\dagger\mathsf{R}^{\dagger-1}[p']^{\dagger-1})D^S_{\nu\tau}([p']^\dagger[\Pi p]^{\dagger-1})D^S_{\tau\sigma}([\Pi p]^{-1}\Pi[p])^{\dagger-1}\\
=&\sum_{\tau} D^S_{\sigma\tau}([\mathsf{R}p']^\dagger\mathsf{R}[\Pi p]^{\dagger-1})\left(D^S([\Pi p]^\dagger)D^S(\Pi)^{\dagger-1}D^S([p]^{\dagger-1})\right)_{\tau\sigma}\\
=&\sum_{\tau} D^S_{\sigma\tau}([\mathsf{R}p']^\dagger\mathsf{R})\left(D^S(\Pi)D^S([p]^{\dagger-1})\right)_{\tau\sigma}\\
=&\, D^S_{\sigma\sigma}([p]^\dagger\mathsf{R}\Pi[p]^{\dagger-1})
\end{aligned} \tag{147}
$$

where the constraint $\mathsf{R}p' = p$ has been used as well as the unitarity of the Wigner rotation and of the matrices $D^S([\Pi p]^{-1}\Pi[p])$ and $D^S(\Pi)$. Therefore, taking (147) into account, the eq. (146) can be simplified into:

$$
\langle p,\sigma|\mathsf{P}_{PJ\lambda}\hat{\Pi}\mathsf{P}_V|p,\sigma\rangle = \delta^4(P-p)\int \mathrm{d}^3\mathrm{p}'\ (2J+1)\int \mathrm{dR}\ D^J_{\lambda\lambda}(\mathsf{R}^{-1})\delta^3(\mathsf{R}\mathbf{p}'-\mathbf{p})
$$
$$
\times\frac{1}{2}\left(D^S([p]^{-1}\mathsf{R}\Pi[p]) + D^S([p]^\dagger\mathsf{R}\Pi[p]^{\dagger-1}\right)_{\sigma\sigma} F_V(\mathbf{p}+\mathbf{p}')\langle 0|\mathsf{P}_V|0\rangle \tag{148}
$$

If we now sum over polarization states σ, the last factor in previous equation yields the trace of $D^S(\mathsf{R}\Pi)$. Since:

$$
\mathrm{tr}D^S(\mathsf{R}\Pi) = \mathrm{tr}\left[D^S(\mathsf{R})D^S(\Pi)\right] = \mathrm{tr}\left[D^S(\mathsf{R})\eta\, I\right] = \eta\,\mathrm{tr}D^S(\mathsf{R})\ . \tag{149}
$$

we finally get:

$$
\begin{aligned}
\sum_\sigma \langle p,\sigma|\mathsf{P}_{PJ\lambda}\hat{\Pi}\mathsf{P}_V|p,\sigma\rangle =&\, \eta\,\delta^4(P-p)(2J+1)\int \mathrm{dR}\ D^J_{\lambda\lambda}(\mathsf{R}^{-1})\\
&\times F_V(\mathbf{p}+\mathsf{R}^{-1}\mathbf{p})\mathrm{tr}D^S(\mathsf{R})\langle 0|\mathsf{P}_V|0\rangle\ .
\end{aligned} \tag{150}
$$

This is, up to a factor η, exactly the same result one would obtain without space inversion, the only difference being in the argument Fourier integral which is now $(\mathbf{p}+\mathsf{R}^{-1}\mathbf{p})$ instead of $(\mathbf{p}-\mathsf{R}^{-1}\mathbf{p})$. Thus, it is not difficult to realize that the contribution to the microcanonical weight of a generic channel with N different particles, of the space-reflected term in the right-hand side of (134) coincides with eq. (61) up to a factor $\Pi \prod_{j=1}^{k} \eta_j^{N_j} \equiv \Pi\Pi_{\{N_j\}}$ and a "plus" sign in the argument of Fourier integrals. Finally, we quote the full expression of the microcanonical channel weight for a spherical system at rest in the Boltzmann statistics:

$$
\begin{aligned}
\Omega_{\{N_j\}} = & \frac{1}{4\pi} \int_0^{4\pi} \mathrm{d}\psi \, \sin\frac{\psi}{2} \sin\left(J+\frac{1}{2}\right)\psi \left[\prod_{j=1}^{K} \frac{1}{N_j!} \prod_{n_j=1}^{N_j} \int \mathrm{d}^3 \mathrm{p}_{n_j} \right. \\
& \times \frac{1}{2} \left(F_V(\mathbf{p}_{n_j} - \mathsf{R}_3(\psi)^{-1}\mathbf{p}_{n_j}) + \Pi\Pi_{\{N_j\}} F_V(\mathbf{p}_{n_j} + \mathsf{R}_3(\psi)^{-1}\mathbf{p}_{n_j}) \right) \\
& \times \left. \left[\frac{\sin(S_j+\frac{1}{2})\psi}{\sin(\frac{\psi}{2})} \right] \right] \delta^4 \left(P - \sum_{n=1}^{N} p_n \right) \langle 0|\mathsf{P}_V|0\rangle
\end{aligned}
\tag{151}
$$

which can be obtained by removing all permutations but identity in eq. (96) and adding the parity-transformed term. The microcanonical partition function can be obtained by summing (151) over all possible channels.

11 Conclusions

We have studied the microcanonical ensemble of a multi-species ideal relativistic quantum gas fixing the maximal set of observables pertaining to the space-time symmetry group. In the rest frame of the system, where $\mathbf{P} = 0$ this means enforcing the conservation of energy-momentum, total angular momentum and parity. We have provided a full consistent treatment and solved the problem for the most general case of particles with spin, generalizing the results of Cerulus [2].

The microcanonical partition function Ω has been expressed as a sum over channels, i.e. at fixed multiplicities of each particle species, each term being defined microcanonical channel weight. These have been obtained in a quantum field theoretical framework, applying to spinorial fields the formulae derived for scalar fields in our previous work [1]. This extension relies on two reasonable, yet unproved assumptions:

(1) the validity of eqs. (47) and (48) which is conjectured on the basis of the $V \to \infty$ limit and the fact that they have been proved for the scalar field [1];
(2) that also for particles with spin, the expressions of the channel weights involving identical particles are the same as those with different particles with (anti)symmetrization, that has been proved for the scalar field [1].

The implementation of angular momentum conservation leads to an expression of the microcanonical weight of a channel involving a further integration over the parameters of the SU(2) group, with respect to the case of only energy-momentum conservation. This expression nicely agrees with that obtained with an expansion onto the angular momentum basis and leads to the expected classical limit, already known and used in literature. We have been working out the grand-canonical partition function with fixed angular momentum in both the case of small and large angular momentum. We also obtained the microcanonical partition function of a system with fixed parity.

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